ปริศนา*

A Study of a Thai Number Puzzle: Two Representations

David Moore, Lindsay Schroeder, Sarah Tomasic,
Eric Wudtke, Kate Ziegelgruber.
Directed by Dr. Louis Smogor.

Abstract

In his paper we constructed two representations and a solution for a common Thai number puzzle. The two representations depicted in our report are a purely mathematical abstraction and the other a more visual geometric representation of the game. Both define the same rules and theorems except in different means. The abstract defines sequences of vertices while the geometric defines certain line segments and points. Upon considering both representations, theorems and corollaries were derived to illustrate methods of solving the puzzle.

Key words: *puzzle, Jordan Curve Theorem, symmetry, sequences*

Abstract Representation

The abstract is not defined as a visual image but rather a purely mathematical representation of the puzzle. Each component of the puzzle is broken down into the simplest form starting with vertices that can be connected by subsequences in order to fulfill the goal of the puzzle. Rules can be delineated as to which subsequences qualify as part of the solution omega. Numbers contained in cells that are defined by four vertices define a unique subsequence of those vertices.

A ปริศนา is an ordered triple (B, N, φ) where

(1) B is a doubly-subscripted set of elements: $B = \{a_{ij}\}$, $1 \leq i \leq m - 1$, $1 \leq j \leq n - 1$. We call B the game board and note that the board defines a number of sets $\{a_{ij}, a_{i\,j+1}, a_{i+1\,j}, a_{i+1\,j+1}\}$ which we call cells; the collection of all cells is referred to as C. Any cell may be identified by the subscripts of the lowest values, so a cell will be called C_{ij}. The elements of the set B are called **vertices** $[a_{ij}]$.

(2) N is a non-empty subset of C.

(3) φ is a function on N into $\{0, 1, 2, 3\}$. The cells in N are said to be numbered; mathematically (2) and (3) tell us $\exists\, N \subseteq C$, $N \neq \varnothing$, and a function $\varphi: N \rightarrow \{0, 1, 2, 3\}$, *i.e.* φ is a function on N, a non-empty subset of C, that takes its values from the set of integers $\{0, 1, 2, 3\}$.

(4) N must exhibit vertical, horizontal or central symmetry.

i. N is said to be **centrally symmetric** iff $\forall\, C_{ij} \in N$ (for all cells in N) there is a numbered cell $C_{(m-i)+1\,(n-j)+1}$ belonging to N.

*Pronounced (bprit-naa).

ii N is said to be **vertically symmetric** if $\forall\ C_{i,j} \varepsilon\ N$ (for all cells in N) there is a numbered cell $C_{(m-i)\ 1\ j}$ belonging to N.

iii N is said to be **horizontally symmetric** if $\forall$ $C_{i,j} \varepsilon\ N$ (for all cells in N) there is a numbered cell $C_{i\ (n-j)+1}$ belonging to N

Definition: Cell C_{ij} is said to **contain** q iff $\varphi(C_{ij}) = q$.

Definition: The cells $C_{1\,j}$, $C_{i\,1}$, $C_{m\,j}$, or $C_{i\,n}$ are said to be **on the border** of B.

Definition: Vertices $a_{i,j}$ and $a_{k\,l}$ are **adjacent** iff $|i - k| + |j - l| = 1$, *i e,* in the index of the vertices, one index differs by one from the other while the other index is the same.

Definition: Cells $C_{i,j}$ and $C_{k\,l}$ are **adjacent** iff $|i - k| + |j - l| = 1$, *i e.* the two cells share only two vertices.

Theorem A1: The total number of cells $|\ C\ |$ is equal to (m-1)(n-1); $|\ B\ |$ = m·n

(5) The goal of ปริศนา is to create a sequence of vertices $\{\Omega_i\}_{i=1}^{k}$ such that

1.) $\Omega_1 = \Omega_k$, *i e*, the first vertex is equal to the last vertex.
2.) If $\Omega_i = \Omega_j$ then $I = j\ \ \forall\ \ 2 \le i, j \le k - 1$, *i.e.* such $a_{i,j}$ can only occur once in Ω_i.
3.) If $\Omega_i = a_{jl}$ then Ω_{i+1} must be one of $a_{j\ 1\,l}$, $a_{j\,l-1}$, $a_{j-1\,l}$ or $a_{j\,l-1}$ *i.e.* Ω_{i+1} must be adjacent to Ω_i.
4.) The following conditions hold:
   a. $\forall\ C_{i,j}\ \varepsilon\ N$ for which $\varphi(C_{i,j}) = 0$, $\exists$ no adjacent pairs of $C_{i,j}$ appear as a subsequence in $\{\Omega_i\}$.
   b. $\forall\ C_{i,j}\ \varepsilon\ N$ for which $\varphi(C_{i,j}) = 1$, $\exists$ a unique subsequence of $\{\Omega_i\}$ that consists of a pair of adjacent vertices of $C_{i,j}$
   c. $\forall\ C_{i,j}\ \varepsilon\ N$ for which $\varphi(C_{i,j}) = 2$, $\exists$ a subsequence $\{\Omega_i\}$ containing exactly three of the vertices of $C_{i,j}$ *or* all four vertices appear in $\{\Omega_i\}$ as two disjoint subsequences.
   d. $\forall\ C_{i,j}\ \varepsilon\ N$ for which $\varphi(C_{i,j}) = 3$, $\exists$ a subsequence of $\{\Omega_i\}$ that consists of the 4 vertices of $C_{i,j}$.

Definition: An **edge** is defined as a subsequence of $\{\Omega_i\}$ of length two. $\{\Omega_i\}$ defines a sequence of adjacent edges. Considering this, we see:

1.) The sequence of edges is closed.
2.) Three edges cannot meet at single point, that is to say the sequence of edges is simple.
3.) Consecutive edges are adjacent.
4.) The following conditions hold.
   i. The sequence contains no edges for cells $C_{i,j}$ containing a 0.
   ii. The sequence contains exactly one edge for cells $C_{i,j}$ containing a 1.

iii. The sequence contains exactly two adjacent edges or exactly two nonadjacent edges for cells $C_{i,j}$ containing a 2.

iv. The sequence contains exactly three adjacent edges for cells $C_{i,j}$ containing a 3.

Remark: The set of edges and vertices so defined constitutes a graph in the graph-theoretical sense of the word.

Geometrical Representation

The geometrical representation of the game is based on visual geometry, beginning with vertices. Vertices are arranged in the game board – a subset of the Cartesian plane indicated by Ψ – and are connected by edges, with each edge containing two vertices. Groups of four vertices that form a square of smallest area is called a cell, with edges equal to the number placed in that cell. The edges are used to form line segments that complete the solution set, β, which solves the puzzle. The solution set (β) always forms a simple closed polygon, so the solution to the puzzle satisfies the Jordan Curve Theorem. β is subject to several important rules that will be discussed below. In all the geometrical representation allows the game to be described visually.

The **board** is a subset of the Cartesian plane

$\Psi = \{ (i, j) \mid i, j \text{ integers}, 1 \le I \le m, 1 \le j \le n, \text{ integers}\}$.

The points of Ψ are **vertices**. These points start out at (1,1). They are lattice points of the plane arranged into columns and rows. There are m columns and n rows.

Any of one of the smallest squares (in terms of area) that can be formed from the points in Ψ is called a **cell**, C.

Remark: No points of Ψ exist in the interior of a cell.

Remark: A cell is determined by the points (i,j), $(i + 1, j)$, $(i, j + 1)$, $(i + 1, j + 1)$, such that $1 \le i < m$, $1 \le j < n$ and contains no other points of Ψ.

Remark: A cell may be unambiguously referred to by its lower left hand vertex (i, j) and will be called C_{ij}.

There is a subset $\boldsymbol{N}$ of the set of cells, $\boldsymbol{C}$, as previously defined.

N must exhibit at least vertical, horizontal or central symmetry and the definitions of these symmetries are as stated before.

Definition: An **edge** is one of the line segments of length 1 joining two distinct vertices of the solution β.

Definition: If two vertices determine an edge, they are said to be **adjacent**

Remark: An edge is a line segment of the form: $\overline{(i,j)(i+1,j)}$ or $\overline{(i,j)(i,j+1)}$.

Theorem G1. There are $(m-1)(n-1)$ cells determined by Ψ.

Theorem G2. There are mn vertices determined by Ψ.

Theorem G3. There are $(n-1)m + (m-1)n$ edges determined by Ψ.

David Moore's Theorem. $2mn - n - m$ is the maximum number of total potential edges that could appear in a solution.
$4+3(m-2)+(3+2(m-2))(n-2)$.

Proof: The number of cells in a row, defined in terms of the number of vertices, is (n -1). The number of cells in column, defined in terms of the number of vertices, is (m -1). The maximum number of lines in a puzzle can therefore be calculated in the following way.

Start in the top left corner with the four lines around that one cell, so we start with 4 lines. Because one side of each neighboring cell in that top row shares one side with the preceding cell, each cell will be considered to have three lines around it. Since the number of cells in a row is (n - 1) and we already counted the first cell, the number of lines in the top row is 3(n - 2). We add this to 4 from the previous step to get 4 + 3(n - 2).

Starting in the first cell of the next row, we have a cell with 3 lines around it, because one side is shared by the cell already counted above it. We therefore start with 3 in each row all the way down the table. With each following cell, we can only count two lines, because one is shared with the preceding cell and the cell above it. Therefore the number of lines around the cells in every row but the top row, excluding the first cell in each row, is 2(n - 2). So, in adding this to the first cell, the number of lines around all the cells in any row but the top row is 3 + 2(n - 2).

Because the number of rows in a puzzle is (m - 1), and we've already counted the first row, the number of remaining rows is (m - 2). So to calculate the number of lines in all the remaining rows we simply multiply the number of rows by the number of lines in each row but the top. This gives us (3 + 2(n - 2))(m - 2).

Adding all of the steps together gives us $4+3(m-2)+(3+2(m-2))(n-2)$, which reduces to 2mn - m - n.

ปริศนา then consists of a board and numbered cells. A solution is a set β of edges such that:

i) Any one edge has a non-empty intersection with exactly two other edges.

ii) The intersection of any three edges must be empty.

iii) For any pair of edges e, f in β, there exists a sequence of edges in β whose first element is e and whose last element is f such that any consecutive pairs of edges in the sequence share a vertex. This causes β to form a simple closed polygon.

iv) If a cell is numbered by k then β includes exactly k edges bordering the cell.

Remark: The Jordan Curve Theorem applies to β.

Let c be a simple closed curve (i.e. a Jordan curve) in the plane $\mathbf{R}^2$. Then the complement of the image of c consists of two distinct connected components. One of these components is bounded (the interior) and the other is unbounded (the exterior). Also, c is the boundary of each component (Wikipedia[1]).

In the gameboard, composed of $(m - 1)(n - 1)$ cells, a cell C_{ij} will be **inside** the closed polygon β if there are an odd number of edges of $\{\beta\}$ in the same row or column of cells above, below, to the right, or to the left of the center C_{ij}. If an even number of edges of $\{\beta\}$ occurs above, below, to the right or to the left in the same row or column as the center of the cell C_{ij}, that cell is considered to be **outside** of the closed polygon β.

The cells $C_{1,j}$, $C_{i,1}$, $C_{m,j}$, or $C_{i,n}$ are said to be **on the border**.

Theorem G4. Line segments must be consecutive. *i.e.* they are all connected.

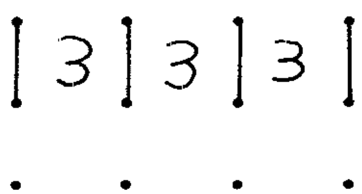
Theorem G5. No three line segments can join at one vertex.

Theorem G6. The solution must have either two or no line segments per vertex.

Note: Having presented the puzzles both abstractly and geometrically and seen that they are equivalent, the forthcoming theorems will be defined in the most convenient manner.

Theorem 7. The solution defines a simple closed polygon. Note, a simple closed polygon is a figure (usually, a plane rectilineal figure) having many, *i.e.* (usually) more than four, angles (and sides); a many-sided figure (OED)[2].

Theorem 8. If C_{ij}, $C_{i+1,j}$, $C_{i+2,j}$, ..., $C_{i+r,j}$ each contain a 3, then $\{a_{ij}, a_{i,j+1}\}$, $\{a_{i+1,j}, a_{i+1,j+1}\}$, ..., $\{a_{i+r,j}, a_{i+r,j+1}\}$, $\{a_{i+r+1,j}, a_{i+r+1,j+1}\}$ are subsequences or their reverses of $\{\Omega_l\}$. *i.e.*, when there is a series of n threes horizontally adjacent to each other, then you can infer $n + 1$ line segments between and outside of the threes.



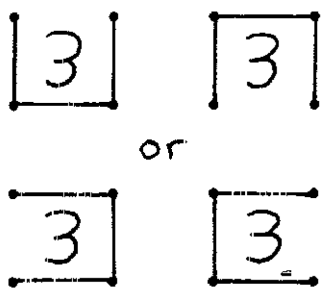
[1] http://en.wikipedia.org/wiki/Jordan_curve_theorem

[2] http://dictionary.oed.com/cgi/entry/50183236?single=1&query_type=word&queryword=polygon&first=1&max_to_show=10

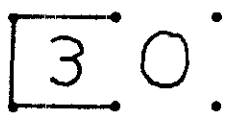
<u>Definition:</u> Cells C_{ij} and C_{kl} are **adjacent** if $|i\text{-}k| + |j\text{-}l|=1$

<u>Theorem 9</u>. If C_{ij} contains a 3, one of the following four subsequences of $\{\Omega_i\}$ or their reverses must be in the solution$\{\Omega_i\}$:

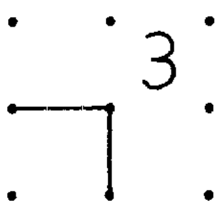
$\{a_{i+1,j+1}, a_{i,j+1}, a_{ij}, a_{i+1,j}\}$, $\{a_{i,j+1}, a_{ij}, a_{i+1,j}, a_{i+1,j+1}\}$, $\{a_{ij}, a_{i,j+1}, a_{i+1,j+1}, a_{i+1,j}\}$, $\{a_{i,j+1}, a_{i+1,j+1}, a_{i+1,j}, a_{ij}\}$



<u>Theorem 10</u>. If C_{ij} contains a 3 and $C_{i+1,j}$ contains a 0, the following subsequence of $\{\Omega_i\}$ or its reverse must be contained in the solution $\{\Omega_i\}$: $\{a_{i+1,j+1}, a_{i,j+1}, a_{ij}, a_{i+1,j}\}$. *i.e.* whenever a three is adjacent to a zero, then there is only one possibility for the three subsequent line segments.



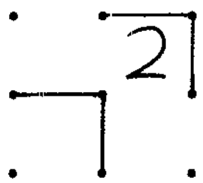
<u>Theorem 11</u>. If C_{ij} contains a 3, and one vertex of a subsequence of three of $\{\Omega_i\}$ is a vertex of C_{ij}, while the other two vertices of the subsequence are vertices of another cell that is not C_{ij}, then the subsequence is not part of $\{\Omega_i\}$.



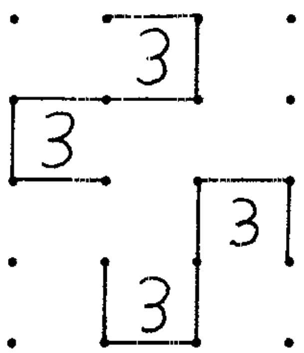
<u>Theorem 12</u>. When there is a two between two zeros, [0 2 0], then the only two line segments possible are below and above the 2.

Theorem 13. On any vertex of a cell containing a three, the cells must be in the interior of any angle formed on the vertices.

Theorem 14. If the polygon already runs through only one vertex around a two, then the polygon has to connect the three remaining vertices.



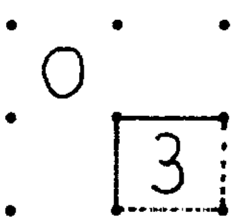
Theorem 15. If two cells containing a three share only one vertex, then edges have to run through the three remaining vertexes on each three



Theorem 16. A number is on the inside of the polygon if there are an odd number of lines above or below the number. Consequently, a number is on the outside if there is an even number of lines above or below the number.

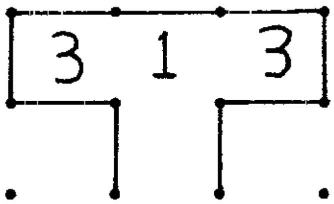
Theorem 17. In the solution, there has to be an even number of edges in each row and in each column.

Theorem 18. If $C_{i(j+1)}$ contains a 0 and $C_{(i+1)j}$ contains a 3, $\{a_{(i+1)j}, a_{(i+1)(j+1)}\}$ and $\{a_{(i+1)(j+1)}, a_{(i+2)(j+1)}\}$ are subsequences of $\{\Omega_i\}$. *i.e.* when you have a 0 diagonal from a three such that the two cells share one vertex there must be two lines running from that shared vertex to the other two vertices on the 3.



Theorem 19. If a cell $C_{i(j-1)}$ contains a 3 and a cell $C_{(i+1)j}$ also contains a 3, the subsequence $\{a_{(i+1)j}, a_{(i+2)j}, a_{(i+2)(j+1)}\}$ and the subsequence $\{a_{i(j+1)}, a_{i(j+2)}, a_{(i+1)(j+2)}\}$, ie when you have two threes diagonal from each other such that they share one vertex the two vertices opposite the shared one have two line segments each coming from that vertex and surrounding the threes

Theorem 20. If you have the sequence [313] against a wall the lines must create a perimeter around the sequence with an opening on the side of the 1 not against the wall.



Theorem 21. For the puzzle to be a closed polygon, there must be an even number of subsequences of $\{\Omega_i\}$ connecting two adjacent vertices.

Theorem 22. If a 0 is located in cell $C_{i(j-1)}$ and a 3 is located in cell $C_{(i+1)j}$ there will be two subsequences of $\{\Omega_i\}$ connecting adjacent vertices: $a_{(i-1)j}$ to $a_{(i-1)(j-1)}$ and $a_{(i+1)(j-1)}$ to $a_{(i+2)(j-1)}$

Theorem 23. If a 3 is located in cell $C_{i(j-1)}$ and another 3 is found in cell $C_{(i-1)j}$ there will be two subsequences of $\{\Omega_i\}$ connecting all vertices of cells $C_{i(j-1)}$ and $C_{(i-1)j}$ except vertex $a_{(i+1)(j-1)}$

Theorem 24. If a cell in the first row or column or the m^th^ row or n^th^ column contains a 3, and adjacent cells contain a 1 and a 3, so that cell $C_{i,j}$ contains a 3, cell $C_{(i+1),j}$ contains a 1, and cell $C_{(i+2),j}$ contains a 3 or cell $C_{i,j}$ contains a 3, cell $C_{i,(j+1)}$ contains a 1, and cell $C_{i,(j+2)}$ contains a 3, every vertex of the cells is part of a subsequence of $\{\Omega_i\}$, except for subsequences $\{a_{(i+1)(j+1)}, a_{(i+1)(j+2)}\}$ and $\{a_{(i+1)(j+1)}, a_{(i+2),(j+1)}\}$ (or their reverses).

Theorem 25. Since the most basic form of Ω_i has an even number of line segments (four vertices), each additional cell C added to the original Ω_i in effect adds two vertices making an additional two subsequences of Ω_i.

Theorem 26. The maximum number of vertices in the closed polygon with sides of length *m* and *n* is *m n*.

Corollary 27: If the total number of vertices is odd, the closed polygon will not have the maximum number of edges m n, it will be m·(n-1). One vertex will not be a part of the closed polygon.

We raise the following questions that might be answered upon further study of ปริศนา:

i Does the puzzle have to be laid out on a flat plane surface?
ii Does the playing board have to be a rectangle?
iii Can we have triangular cells instead of square cells?
iv Can we have a prime number of dots?
v Do we have to have an even number of dots?
vi Do they have to be evenly distributed?