Graphical Solutions to Instant Insanity Puzzles, Combinatorics of These Puzzles, and Their Generalizations

Catie Baker

Sara Drury

Rachel Rominger

Matt Keinsley

Yi Wang

Section 1: Introduction

We discovered Catie graphs, which will be discussed later, when attempting to solve the Instant Insanity puzzle. The Instant Insanity puzzle has four cubes where each face of each cube has one of the four colors, red, green, white, or blue. The objective is to stack the four cubes on top of one another so that when looking at a column where each cube has one face showing the column should have a different color for every face.

Section 2: Important Definitions

In order to understand the concepts we will be discussing, the following definitions will be helpful.

Recall that a **graph** is a set of ordered triples (V, E, ϕ), where V is a finite non-empty set, E is a finite set, and ϕ is a function such that ϕ: $E \to S$ where $s \in S$ iff $s \subseteq V$ and $0 <\mid s \mid \leq 2$. An element in the set V is called a **vertex** and the elements in set E are the **edges**. This definition of a graph allows for multiple edges and loops. If there are no multiple edges and no loops, the graph is said to be a simple graph.

A **loop** is an edge whose image is a single element set. Thus, if $\mid \phi(e) \mid = 1$, it is a loop.

The edges, $e_1$and e_2 are said to be **multiple edges** when the image of one edge is equal to another edge. So, if $e_1$and $e_2 \in E$ and $\phi(e_1) = \phi(e_2)$, then they are considered multiple edges.

The **endpoints** of an edge e are the vertices in the image of the function $(\phi(e))$.

Remark 1: An edge can only have one or two end points.

An **equivalence class** is a subset of the sets on which a relation is defined when the relation is symmetric, transitive, and reflexive. If the relation on a set, S, is denoted as $\sim$, then it is **symmetric** if $a \sim b$ implies $b \sim a$, **transitive** if $a \sim b$ and $b \sim c$ implies $a \sim c$, and **reflexive** if $a \sim a$ for all $a \in S$. We know that x is in the same equivalence class as y iff $x \sim y$.

A **subgraph** of the graph (V, E, ϕ) is a graph with the ordered triple (V_1, E_1, ϕ_1) such that $V_1 \subseteq V$, $E_1 \subseteq E$, and $\phi_1 \subseteq \phi$. Two subgraphs are **disjoint** if the intersection of their edges is an empty set.

Labeling is a function from the set of edges to another set, called the set of labels.

A **partition** of a set is a set of subsets of the set where the subsets are non-empty, the union of all subsets is the original set, and the subsets are mutually (pair-wise) disjoint. An equivalence relation partitions the set on which it is defined.

Recall that an edge, e, is said to be **incident** on vertex, v, iff $v \in \phi(e)$. An edge, e, is **incident** on a vertex v if v is an endpoint of e.

Also recall that **degree** of a vertex is the number of edges incident on that vertex, with loops being counted twice.

Two graphs, G_1 and G_2, are **isomorphic** iff there exists a 1-1 function, φ, going from the set of vertices for G_1 onto the set of vertices of G_2 such that for every edge, vw where $v, w \in V_1$, in G_1 there is a corresponding edge in G_2 $\varphi(v)\varphi(w)$.

A graph is **planar** if no edges cross over each other.

Section 3: Solving The Instant Insanity Problem

With the definitions above, we are going to solve the Instant Insanity Problem, applying this knowledge of graphs and relations.

Theorem 1: In Instant Insanity, a cube is an equivalence class of ordered triples, each element of which is an equivalence class of ordered pairs of colors.

Proof: To define the cube, only three pieces of information are needed, the sets of the colors of the opposite faces. We know this is true since, if you were to exchange the color of a pair of opposite faces, the resulting pair of faces would still be bordered by faces of the same four colors as it was earlier. Thus, if you are given the colors of three pairs of opposite faces, you will always make the same cube. We know that the pairs of opposite faces can be represented using an equivalence class of ordered pair of colors. We know that it needs to be an equivalence class as a pair of sides designated as (Red,Green) is equivalent to the pair designated (Green,Red). The order of the sets of opposite sides does not matter; the entire cube is represented as an equivalence class of ordered triples, with each element being an equivalence class of ordered pairs. □

Every cube can be represented as a graph. The graph of the cube would have the set of the four colors as vertices. Each equivalence class of ordered pairs would represent an edge where the endpoints of the edge are the elements of the ordered pair. So if a cube has an equivalence class of ordered pairs $(Red, Green)$, the edge has endpoints at the red vertex and the green vertex. Every edge will have the same label representing that they came from the same cube. A **Catie**

graph is is the graph that combines the 4 graphs of the cubes. It will have 4 vertices which are labeled with the colors, 12 labeled edges such that the set of edges is partitioned by labels into subsets of cardinality 3.

Theorem 2: The solution to the Instant Insanity Problem can be represented graphically with two disjoint subgraphs such that each subgraph has 4 vertices and 4 edges such that the each vertex has degree 2 and every edge in the subgraph has a different label.

Proof: Since the cubes can be represented as a series of equivalence class of ordered triples, $\overline{(P_{r1}, P_{r2}, P_{r3})}$where $r \in \{1, 2, 3, 4\}$ is the cube represented and P is the equivalence class of ordered pairs defining the cube. To find a solution to the Instant Insanity Puzzle, you must first find a quadruple, $\{P_{1\mathrm{i}}, P_{2\mathrm{j}}, P_{3\mathrm{k}}, P_{4\mathrm{l}},\}$ where the first subscript of P represents the ordered triple the pair comes from and i, j, k, l are such that $0 < i, j, k, l \leq 3$ and represents which equivalence class of ordered pairs is taken from each ordered triple. The elements from the ordered pairs need to be able to be partitioned into 2 sets of cardinality 4. These sets must not have two sides from the same cube, so the first subscript must be unique. Once the first quadruple has been created, a second similar quadruple must be found from the remaining 8 edges. This can be represented graphically by two disjoint subgraphs, with the subgraph representing the quadruple. The subgraph will have four edges and four vertices such that each vertex has degree two and each label is only used once per subgraph. Two disjoint subgraphs are needed since no face of the cube can be used in more than one place. Each subgraph represents a column of faces and the column opposite it. Since there are four edges, that allows for one edge to come from each cube, which is what is specified by the fact that each label is only used once and that the first subscripts must be different. The four vertices represent the four colors that must be included. Every vertex must have degree two since for a set of two columns, we need two faces with each color. □

Remark 1: Solutions are not necessarily unique

To create a another, distinct, solution, you must be able to move the cubes such that the faces showing for at least two cubes are different than the faces showing on the original solution. We can see that this is possible by looking at the chart below. In the chart below $C_1, C_2, C_3,$ andC_4 are the columns that are the visible faces of the cubes. C_1andC_2 is one set of opposite faces and C_3 and C_4 are the other. The faces not visible are H_1 and H_2. $\mathbb{C}_1$, $\mathbb{C}_2$, $\mathbb{C}_3$, and $\mathbb{C}_4$ represent the cubes. The below chart represents a graph that has a solution since in each column C_i, every color is represented once. The Xs can be any color as they do not affect the solution.

	C_1	C_2	C_3	C_4	H_1	H_2
$\mathbb{C}_1$	B	R	R	G	G	G
$\mathbb{C}_2$	G	B	G	B	R	B
$\mathbb{C}_3$	W	W	B	R	X	X
$\mathbb{C}_4$	R	G	W	W	X	X

We can see that this example has a different solution by changing the faces showing for the C_3/C_4 set of opposite sides making the solution look like this:

	C_1	C_2	C_3	C_4	H_1	H_2
$\mathbb{C}_1$	B	R	G	G	R	G
$\mathbb{C}_2$	G	B	R	B	G	B
$\mathbb{C}_3$	W	W	B	R	X	X
$\mathbb{C}_4$	R	G	W	W	X	X

In this new solution, we can see that cubes 1 and 2 have different faces showing, but each column still has every face represented once. Thus there is a second, distinct, solution that can be created from this set of cubes. $\square$

Remark 2: It is not necessary for the ordered quadruples to be distinct to create a solution.

An example of this would be:

	C_1	C_2	C_3	C_4	H_1	H_2
$\mathbb{C}_1$	B	R	B	R	X	X
$\mathbb{C}_2$	G	B	G	B	X	X
$\mathbb{C}_3$	W	W	W	W	X	X
$\mathbb{C}_4$	R	G	R	G	X	X

As you can see from this example, the both quadruples would be

$$\{(B,R),(G,B),(W,W),(R,G)\}$$

for both ordered quadruples. $\square$

Theorem 3: The maximum number of solutions that one set of Instant Insanity Cubes can have is 25.

Since we know that there can be multiple solutions created by exchanging sides from the 3rd subgraph, we know that we can either exchange 2, 3, or 4 edges to create a distinct solution. If we have a set of four edges that we can swap, there are 3 different distinct solutions possible as we can choose any two of the three complete subgraphs to be showing. If we swap three edges out from the graphs, there are $\binom{4}{3} = 4$ ways to choose which three edges to swap. Since it is possible that the three edges could be swapped with edges on either graph we must multiply 4 by 2 to get the number of solutions possible with 3 edges swapped out. If we only swap to edges out, there are $\binom{4}{2} = 6$ ways to determine which edges we swap out. Then there is a possibility that the two edges not swapped in the 3rd subgraph could be swapped out in the other subgraph. Thus there is are three different ways that you could swap the two sets of two edges, both sets being swapped or one set or the other being swapped. Thus there are $6*3 = 18$ ways to swap two edges or two sets of two edges out. Once we have all the different possibilities of how we can swap edges out, we can see that there are $3+8+18 = 25$ possible solutions a set can have.

Section 2: Combinatorics Relevant to the Instant Insanity Problem

Lemma 1: There are 10 different sets of opposite sides possible.

Proof: We know that we can either have a set of opposite faces that has two different colors or both of the opposite faces have the same color. There are four possible sets of opposite faces if both faces are the same color, one set for each color. If the opposite faces are different colors, there are 6 possibilities since we want to choose 2 colors from 4 and $\binom{4}{2}$equals 6. Thus there are 10 possible sets of opposite faces. □

Lemma 2: There are 220 different instant insanity puzzle cubes possible.

Proof: Once we know how many possible sets of opposite faces there are, we can use that information to determine how many cubes are possible. In order to account for the possibility of multiple edges in the Catie Graph, we can use dividers to count the number of possible cubes. We know that to divide a group into 10 parts, there must be 9 dividers. Since we know that we have 3 sets of opposite faces, we can determine the number of cubes by determining how many ways it is possible to place the three sets in the dividers. If each divider and each set of opposite faces takes up 1 space, there are 12 spaces that we have to work with. The dividers will always go in the same order, so we only need to choose where to have a divider. Any space that does not have a divider is where the set of opposite sides would be. Thus, since there are 12 spots for the 9 dividers, there are $\binom{12}{9}$ ways to place the dividers, thus there are $\binom{12}{9}$=220 possible cubes. □

Theorem 4: There are 100,290,905 possible Catie graphs.

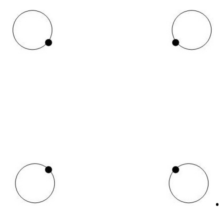
Proof: We can use the same method we used to discover the number of cubes possible to determine how many sets of cubes are possible. There will be 219 dividers, so there are $\binom{223}{4}$ = 100,290,905 possible sets of cubes.

Remark 3: Not every set of four cubes will be solvable.

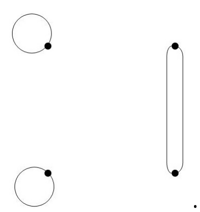
As there are no requirements for the set of cubes other than the fact that it needs to have four cubes with the options of faces from all four colors, we are not guaranteed a solution. The obvious example would be a set of four cubes where each cube was all one color and every cube was the same color. It is obvious that a set of four all red cubes would not have a solution, as every color would not be represented. □

Lemma 3: Up to isomorphism, there are 5 possible subgraphs that have the characteristics required of a solution.

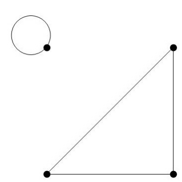
Proof: Since we know that in a solution a vertex can only have degree 2, we know that it will either have a loop or have two edges incident on it. If we start with every vertex having a loop, we find our first subgraph,



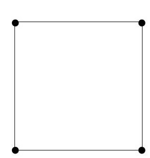
If we then remove one loop, we cannot create another subgraph. This is because we only have one vertex without degree two, thus the only possible way to add an edge and have a vertex with degree two is to add an loop, which is the graph we have already found. If there are only two loops, we have two vertices which the remaining to edges can be incident on. The only way for this to be possible is if the two edges are incident on the two unused vertices, which is subgraph B,



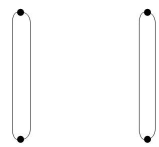
If we only have one loop, then the only way to have a usable subgraph is if there are no multiple edges and this is subgraph C,



If there is a set of two multiple edges, there would be one edge and one vertex left requiring it to be a loop on the vertex, which is a graph we have already found. The last possibility is if there are no loops. In this case there are two subgraphs possible. The first is if there are no multiple edges, subgraph D,



The second is if there are two sets of multiple edges on different vertices, subgraph E,


.$\square$

Theorem 5: There are at most 9,960,300 solvable Catie graphs.

Since we know that there are 5 isomorphic subgraphs, we know that there are 15 possible combinations of two subgraphs to create a solution. We know this since there are two different subgraphs in the solution, there are $\binom{5}{2}$ possibilities and if there are two isomorphic subgraphs there are $\binom{5}{1}$ possibilities. To figure out how many of the sets had solutions, we first determined how many graphs there were for each isomorphic form when you label the vertices. For subgraph A, there was only one form. For subgraph B, there were $\binom{4}{2}$ ways as you would choose two vertices to be the vertices with the loop. For subgraph C there are four forms as there are $\binom{4}{1}$ ways to choose which vertex has the loop. There are $\binom{3}{1}$ ways to have subgraph D. Since there are six edges possible with distinct endpoints and we only use 4 of the edges, we know that there are multiple graphs possible. If we select any of the 4 vertices, we can determine how many possibilities there are for subgraph D by selecting which of the 3 remaining vertices do not have an edge which is incident on the 2 vertices. There are 6 forms of subgraph E as there are $\binom{4}{2}$ ways to choose two vertices to have a pair of multiple edges and the other two vertices will automatically have a pair of multiple edges. An example of a set and its solution for each combination is included in the appendix. Once we know the number of forms each subgraph has, we can determine how many possible sets for each combination of two subgraphs. For each subgraph, there is a basic pattern we used to determine how many sets were possible. First we need to choose which form of the subgraphs we will use. Then we will multiply those numbers by 24 because there are 24 ways the second graph can be labeled (the way the first graph is labeled is irrelevant as there is no real difference between label 1 and 2, they are just there so we know which edges of the second subgraph come from the cubes that are already determined.) Next we will account for any multiple edges by dividing by 2 for each pair of multiple edges. This is because if pair one and two are equivalent, then it would not be a different set if we switched the labels. The chart below show the the number of sets of the possible for the combinations indicated.

	A	B	C
E	$6(24)/(2*2) = 36$	$(6*6*24)/(2*2*2) = 108$	$(4*6*24)/(2*2) = 144$
D	$3*24 = 72$	$(3*6*24)/2 = 216$	$4*3*24 = 288$
C	$(4*24) = 96$	$(6*4*24)/2 = 288$	$4*4*24 = 384$
B	$(6*24)/2 = 7$	$(6*6*24)/(2*2) = 216$	
A	24		

	D	E
E	$(24*3*6)/(2*2) = 108$	$(6*6*24)/(2*2*2*2) = 54$
D	$3*3*24 = 216$	

After finding the number of ways that we could make the 15 different types of solutions, 1,674, we have to determine how many of the third subgraphs would create a set with multiple solutions and make sure that they are only counted once. First we need to determine how many third graphs we have possible. We can determine this by finding out how many third subgraphs have no, 2, 3, and 4 multiple edges or have two sets of two multiple edges. We want to find these separately as it will account for the fact that the labels of multiple edges can be switched without creating a new subgraph. Thus we figure out that for a subgraph with no multiple edges there are $10*9*8*7 = 5,040$ subgraphs. Each number we multiply represents how many options we have for the edges. For the graphs with multiple edges, once we know which vertices the multiple edge will be incident on, we know there is only one option for the other multiple edges. For a subgraph with 2 multiple edges there are $10*9*8 = 720$ possible subgraphs, there are $10*9 = 90$ subgraphs possible when there are 3 multiple edges and 10 possible when there are 4 multiple edges. There are $10*9 = 90$ possible subgraphs with two sets of multiple edges. Thus there are 5,950 third subgraphs possible. So we know that at most there are $1,674*5,950 = 9,960,300$ sets.□

Once we knew the upper bound of the number of sets with solutions, we began to look for a lower bound. Since a set can have up to 25 solutions, we decided to first discover the number of sets with only one solution by subtracting all the sets that have a third subgraph which can be transformed to create a distinct solution. To figure this out, we looked at the different ways to swap edges to figure out how many of the sets have multiple solutions. The chart below, shows how the graphs can be transformed to create a different solution.

$Start\,with$	$Move\,2\,edges$	$Move\,3\,edges$	$Move\,4\,edges$
A	A, B	A, B, C	A, B, C, D, E
B	A, B, C, E	A, B, C, D	A, B, C, D, E
C	B, C, D	A, B, C, D, E	A, B, C, D, E
D	C, D, E	B, C, D, E	A, B, C, D, E
E	B, D, E	C, D, E	A, B, C, D, E

For each type of solution we need to know how many to subtract to account for the multiple solutions. We started with the easiest of the swaps, moving 4 edges. There are 300 different subgraphs that are one of the complete subgraphs.

Next we need to look at the ways to exchange 3 sides. For each of the transformations shown above, we need to determine how many ways there are to choose which 3 edges to swap, how many ways there are to label the 3 edges, and how many possibilities there are for the fourth edge that won't make one of the subgraphs we already accounted for.

$From$	To	$Which\,edges$	$Labels$	$4th\,edge$	$Total$
A	A	4	6	9	216
A	B	4	3	9	108
A	C	4	6	9	216
B	A	2	6	9	108
B	B	4	3	9	162
B	C	4	6	9	216
B	E	2	6	9	108
C	A	1	6	9	54
C	B	4	6	9	216
C	C	4	6	9	216
C	D	3	6	9	162
C	E	3	3	9	81
D	B	4	6	9	216
D	C	4	6	9	216
D	D	4	6	9	216
D	E	4	3	9	108
E	C	4	6	9	216
E	D	4	6	9	216
E	E	4	3	9	108

To determine how many of the third subgraphs to subtract to account for swapping 3 edges, we need to add the total of each of the rows that start with a subgraph of the type that the solution contains. So if the main subgraphs are AB we are going to subtract $(216 + 108 + 216 + 108 + 162 + 216 + 108)$ from 5,950.

Finally we look at the ways we can swap two edges and transform one of the subgraphs into different complete subgraph. Because it is possible to swap 2 edges to one subgraph and 2 edges to the other subgraph, it is much more difficult to determine which edges cannot be included in the third subgraph. If we look at the solution type AB, we can see the difficulty. If the third subgraph has two of the edges that can be swapped with the A subgraph, then to determine which edges not to include, we look at the edges that would make one of the possibilities we already accounted for and we need to look at the edges that would be counted when we looked at the possibilities for exchanging two edges with subgraph B. We have to be careful not to count it twice in our subtraction, but we also have to make sure we count it at least once. The problem is, whether or not the edges that can be swapped with B depends on what labels are left for the other 2 edges. In some cases the other two edges will not matter as the edges that share the label in subgraph B cannot be swapped out. In other cases, the other 2 edges will matter as they could be swapped out to create a different solution using the other subgraph and that subgraph will be subtracted more than once. Due to the many factors that we had to take into account, we were able to figure out everything except how many possibilities there are for the other 2 edges on the third subgraph. What we were able to figure out is displayed in the chart below. Some of the transformations have multiple rows in the chart as there are different ways to transform the subgraphs which cause the number of ways to label the edges to be different.

$Form\, if\, multiple$	$From$	To	$Which\, edges$	$Labels$
	A	A	6	2
	A	B	6	1
	B	A	1	2
$2\, multiple\, edges$	B	B	1	1
$2\, loops$	B	B	1	2
$1\, loop$	B	B	2	2
	B	C	2	2
	B	E	1	1
$2\, loops$	C	B	1	2
$1\, loops$	C	B	2	2
	C	C	6	2
	C	D	3	2
	D	C	6	2
$connected\, edges$	D	D	4	2
$connected\, edges$	D	D	2	2
	D	E	2	2
	E	B	2	2
	E	D	4	2
	E	E	6	2

Once we know how many possibilities there are for the last 2 edges for each of the 15 combinations of solutions, it will be easy to determine how many sets there are with only 1 solution. We just take each of the 15 combinations and multiply the number of sets for each combination by whatever number we get when we subtract the number of possibilities to swap 2, 3, or 4 edges and create a different subgraph from the original subgraph from 5,950.

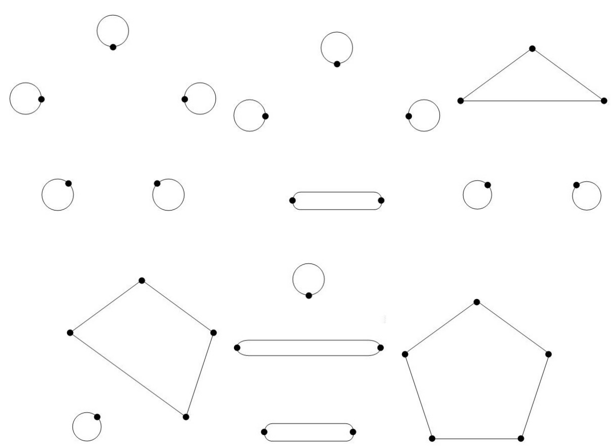
Section 3: Generalized Catie Graphs

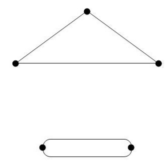
The Catie graph that we have been working with is a (4,4,3,2) Catie graph. The Catie graph can be generalized to a (w, x, y, z) Catie graph, where w is the number of objects, x is the number of distinct labels for the pieces of information, y is the number of pieces of information needed to identify the objects, and z is the number of parts that the piece of information requires. The Catie Graph would have x vertices, wy edges, and w labels. The edges will be partitioned into the labels into sets with equal cardinality, which will be z. We looked at solutions to Catie Graphs when $y = 3$ and $z = 2$. If z was something other than two, we would not be able to represent it with a Catie Graph as the piece of information would no longer be able to be represented as an edge. We only looked at Catie Graphs where $y = 2$ because we would not know how many of the subgraphs would need to fit the criteria we developed. It could always be 2, it could be $y - 1$ or it could be something else.

If we were to examine the graphical solution that was described by a (w, x, y, z) Catie Graph where $w < x$, there would be a small difference from the $(4, 4, 3, 2)$ Catie Graph. The vertices in subgraphs of the solution would not have to have degree 2, but would have a maximum of degree 2 with the total degree of the graph being $2w$ as one edge will come from each object. It would not be possible to have a solution to a (w, x, y, z) Catie Graph if $w > x$. If each vertex has a maximum of degree 2, then there will be a maximum of degree $2x$ for the subgraph. Since there are w objects, there would be degree $2w$ if each edge was used. Since $2x \not\geq 2w$ we can see that there would have to be at least one vertex with a degree greater than two.

Example 1: A (5,5,3,2) Catie Graph

We decided to investigate puzzle sets of a (5,5,3,2) Catie graph. There would be a (5,5,3,2) Catie graph when there are 5 cubes with 5 colors possible. We determined that there were 680 cubes possible. There were fifteen sets of opposite sides possible, which requires fourteen dividers, and three pieces of information were needed to uniquely identify each cube, thus there are . We then determined that it was possible to create 1,229,521,022,136 sets of five cubes using the 680 different cubes. We then determined there were seven possible subgraphs that represent solutions to the cube sets. Since there were five colors represented on each cube, there are five vertices. Further investigation revealed that with 5 vertices, it is possible to create 7 distinct subgraphs that represent "solvable" combinations of cubes. We found the subgraphs in the same manner we found them for a $(4, 4, 3, 2)$ Catie Graph. We started with all loops and then found which new subgraphs you could make every time you subtracted a loop. The subgraphs we found were:

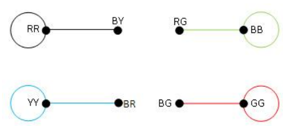


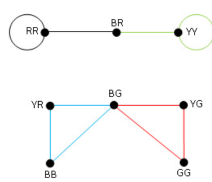


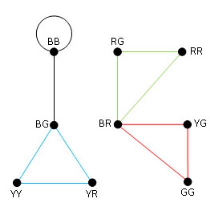
Section 4: Generalized Insanity Graphs

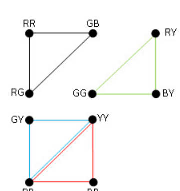
The problem with the generalized Catie graph is that it only works if the pieces of information are in pairs. To overcome this issue, we developed another way of representing the information. In the Catie graph, the information was show in the form of a edge. In the generalized insanity graphs, the vertex is the full piece of information. The edges connect different vertices to represent that they come from the same object.

We created a graphs for each of the 15 combinations of subgraphs to look for patterns in the solutions. The graphs looked like this:

AA: 

AB: 

AC: 

AD: 

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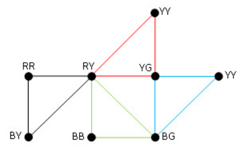

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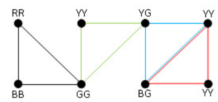

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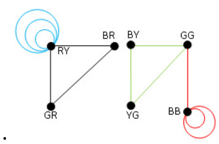

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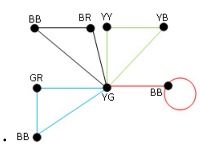

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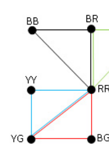

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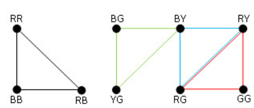
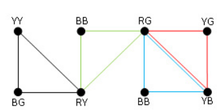
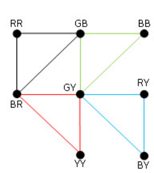
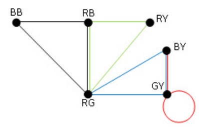


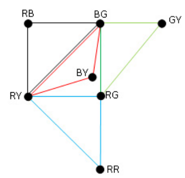
CD:

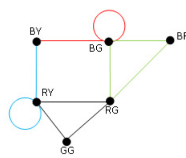


CE:



DD: 

DE: 

EE: 

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From these graphs, there was no clear pattern that emerged to the general form of a graph that has a solution. What we did discover was that every graph we made for a set that had a solution was planar.

Section 5: Further Research

There were many questions that we would have explored had we had more time. One of these areas is the solution to a generalized Catie Graph. We would have liked to be able to look closer at what constitutes a solution when the number of pieces of information is more than 3. Without an concrete examples of this type, we were unsure whether there would always be 2 subgraphs that need to fit the criteria or if it will be based off the number of pieces of information.

We also would have looked for another way to represent the Catie graphs when there are more than 2 pieces of information. We tried to use the Insanity graphs, but these graphs could not be easily used to discover if there is a solution. Also we only looked at the Insanity graphs when there are 3 pieces of information. It would be interesting to investigate the Insanity graphs that have more than 3 pieces of information.

Another thing that we would do if we had more time would be to investigate the number of sets with solutions. We found an upper bound for the number of solutions and we would like to find the lower bound and the exact number of sets with solutions.

Appendix

Below are examples of a solution for each of the combinations of subgraphs:

AA:

	C_1	C_2	C_3	C_4	H_1	H_2
$\mathbb{C}_1$	B	B	R	R	X	X
$\mathbb{C}_2$	R	R	G	G	X	X
$\mathbb{C}_3$	W	W	W	W	X	X
$\mathbb{C}_4$	G	G	B	B	X	X

AB:

	C_1	C_2	C_3	C_4	H_1	H_2
$\mathbb{C}_1$	B	B	B	B	X	X
$\mathbb{C}_2$	G	G	G	R	X	X
$\mathbb{C}_3$	W	W	W	W	X	X
$\mathbb{C}_4$	R	R	R	G	X	X

AC:

	C_1	C_2	C_3	C_4	H_1	H_2
$\mathbb{C}_1$	B	B	B	R	X	X
$\mathbb{C}_2$	G	G	G	B	X	X
$\mathbb{C}_3$	W	W	W	W	X	X
$\mathbb{C}_4$	R	R	R	G	X	X

AD:

	C_1	C_2	C_3	C_4	H_1	H_2
$\mathbb{C}_1$	B	B	B	R	X	X
$\mathbb{C}_2$	G	G	G	B	X	X
$\mathbb{C}_3$	W	W	R	W	X	X
$\mathbb{C}_4$	R	R	W	G	X	X

AE:

	C_1	C_2	C_3	C_4	H_1	H_2
$\mathbb{C}_1$	B	B	B	R	X	X
$\mathbb{C}_2$	G	G	R	B	X	X
$\mathbb{C}_3$	W	W	W	G	X	X
$\mathbb{C}_4$	R	R	G	W	X	X

BB:

	C_1	C_2	C_3	C_4	H_1	H_2
$\mathbb{C}_1$	B	B	B	B	X	X
$\mathbb{C}_2$	G	R	G	R	X	X
$\mathbb{C}_3$	W	W	W	W	X	X
$\mathbb{C}_4$	R	G	R	G	X	X

BC:

	C_1	C_2	C_3	C_4	H_1	H_2
$\mathbb{C}_1$	B	B	B	R	X	X
$\mathbb{C}_2$	G	R	G	B	X	X
$\mathbb{C}_3$	W	W	W	W	X	X
$\mathbb{C}_4$	R	G	R	G	X	X

BD:

	C_1	C_2	C_3	C_4	H_1	H_2
$\mathbb{C}_1$	B	B	B	R	X	X
$\mathbb{C}_2$	G	R	G	W	X	X
$\mathbb{C}_3$	W	W	W	B	X	X
$\mathbb{C}_4$	R	G	R	G	X	X

BE:

	C_1	C_2	C_3	C_4	H_1	H_2
$\mathbb{C}_1$	B	B	B	R	X	X
$\mathbb{C}_2$	G	R	R	B	X	X
$\mathbb{C}_3$	W	W	G	W	X	X
$\mathbb{C}_4$	R	G	W	G	X	X

CC:

	C_1	C_2	C_3	C_4	H_1	H_2
$\mathbb{C}_1$	B	R	B	R	X	X
$\mathbb{C}_2$	G	B	G	B	X	X
$\mathbb{C}_3$	W	W	W	W	X	X
$\mathbb{C}_4$	R	G	R	G	X	X

CD:

	C_1	C_2	C_3	C_4	H_1	H_2
$\mathbb{C}_1$	B	R	B	R	X	X
$\mathbb{C}_2$	G	B	G	B	X	X
$\mathbb{C}_3$	W	W	R	W	X	X
$\mathbb{C}_4$	R	G	W	G	X	X

CE:

	C_1	C_2	C_3	C_4	H_1	H_2
$\mathbb{C}_1$	B	R	B	R	X	X
$\mathbb{C}_2$	G	B	R	B	X	X
$\mathbb{C}_3$	W	W	G	W	X	X
$\mathbb{C}_4$	R	G	W	G	X	X

DD:

	C_1	C_2	C_3	C_4	H_1	H_2
$\mathbb{C}_1$	B	R	B	R	X	X
$\mathbb{C}_2$	G	B	G	B	X	X
$\mathbb{C}_3$	R	W	R	W	X	X
$\mathbb{C}_4$	W	G	W	G	X	X

DE:

	C_1	C_2	C_3	C_4	H_1	H_2
$\mathbb{C}_1$	B	R	B	R	X	X
$\mathbb{C}_2$	G	B	R	B	X	X
$\mathbb{C}_3$	R	W	G	W	X	X
$\mathbb{C}_4$	W	G	W	G	X	X

EE:

	C_1	C_2	C_3	C_4	H_1	H_2
$\mathbb{C}_1$	B	R	B	R	X	X
$\mathbb{C}_2$	R	B	R	B	X	X
$\mathbb{C}_3$	G	W	G	W	X	X
$\mathbb{C}_4$	W	G	W	G	X	X