The first way of thinking that made the law about the behavior of light evident was discovered by Fermat in about 1650, and it is called *the principle of least time*, or *Fermat's principle*. His idea is this: that out of all possible paths that it might take to get from one point to another, light takes the path which requires the *shortest time*.

Richard Feynman

Lifeguard Problem

You are a lifeguard on a beach. You are standing on the shore, the water laps at your feet as the sun shines on your face. You timed yourself the other day and you ran 100 meters in 20 seconds. You are much slower than the other lifeguards, but you are a super-fast swimmer. You can swim 100 meters in 50 seconds—that's only a few seconds off the freestyle world record! You like numbers and a thought pops into your head, "It's funny that I'm a slow runner and fast swimmer, yet I can still run 2.5 times faster than I swim—my run speed is 5 m/sec while my swim speed is 2 m/sec."

Suddenly, you see someone frantically waving in the water and hear cries of "Help!" They are exactly 100 meters to your right and 100 meters from the shore. Figure 2.1 shows the positioning. Your training kicks in and you take off running along the shore towards the drowning person. Your goal is to get to them as quickly as you can. Your mind is consumed by a critical question, "When do I dive in and start swimming?"

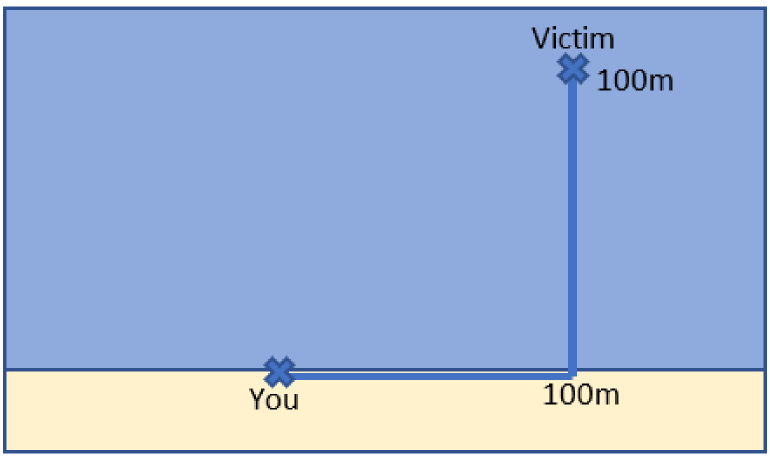

Figure 2.1: Initial Positions of the Lifeguard and Victim.

Before we implement and solve this problem in Excel, let's agree that you do have a decision to make. Consider these two paths:

(A) Based on the logic that you're a faster runner than swimmer, you could run 100 meters along the shoreline in 20 seconds and then swim 100 meters in 50 seconds and you would reach the drowning person in 70 seconds. This path is the shortest swimming distance.

(B) On the other hand, you could argue that it is better to just dive into the water and make a beeline to the victim because this is the shortest total distance. The Pythagorean Theorem says that the hypotenuse is about 141 meters (sqrt($100^2 + 100^2$)). This is much shorter than the 200 meters total distance required by Path A.

The problem, however, is not to minimize swimming distance or total distance, but time—you have to get to the drowning victim as soon as possible. We know Path A takes 70 seconds, but what about Path B?

Path B takes a little longer than A. Here is the algebra involved in the computation:

$$time = \frac{distance}{speed} = \frac{141m}{2\frac{m}{sec}} = 70.5 \text{ seconds}$$

Thus, running 100 meters and swimming 100 meters is better than swimming about 141 meters because 70 seconds is a little less than 70.5 seconds. The advantage of Path B, less distance, is counteracted by the fact that you have to swim more and you are a much slower swimmer than runner. The advantage of a shorter total distance is outweighed by the disadvantage or cost of having to swim more.

You might think that we are done since we figured out that Path A is better than B, but we have just scratched the surface of the lifeguard problem. Figure 2.2 shows that there are not only two paths. You can dive in immediately or run 100 meters and then swim or choose any any distance in between! The problem is to find the best, fastest path.

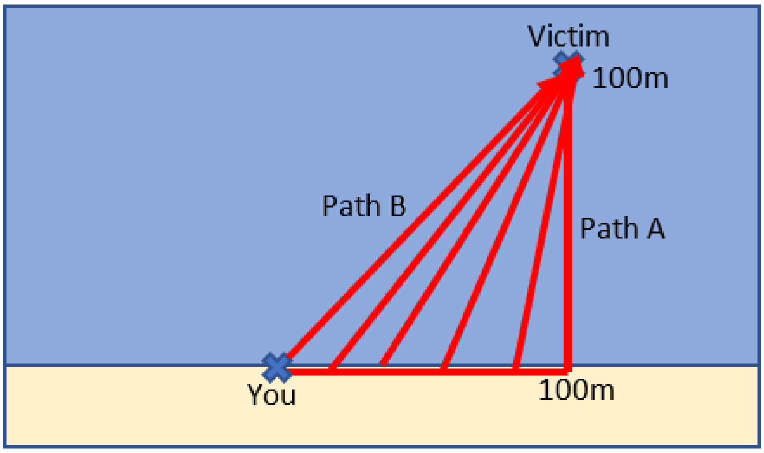

Figure 2.2: Of the many paths, which is the best, fastest path?

The lifeguard problem can be solved analytically, with calculus and algebra, but we will not do that. Instead, we will use *numerical methods*: an algorithm is used to get an approximate result. We will implement the problem in Excel and use Excel's Solver add-in to find the correct answer.

STEP Open a blank Excel workbook and save it as *LifeguardProblem.xlsx*. Enter the numbers and labels shown in Figure 2.3. Make sure your Excel workbook looks exactly like Figure 2.3 because we will be referencing specific cell addresses below.

	A	B	C	D
1	Exogenous Variables			
2	100	Distance Away on Sand, m		
3	100	Distance Away in Water, m		
4	5	Max Run Speed, m/sec		
5	2	Max Swim Speed, m/sec		
6				
7	Endogenous Variables			
8	0	Distance Run, m		
9	141.4214	Distance Swim, m		
10				
11	Goal			
12	70.71068	min Time to Victim, sec		

Figure 2.3: Setting up the problem in Excel.

Exogenous variables cannot be changed by the decision-maker. From the Greek, *exo* means outside, like a lobster's exoskeleton. Exogenous variables are outside or beyond our control. They are also known as independent, given, or constant variables. They are parameters that are part of the decision-maker's environment.

You certainly cannot control where the victim is drowning. If you could, the problem would be trivial—you would put them right at your feet and pull them out. So, the location of the drowning swimmer is exogenous, 100 meters away on the sand and 100 meters from the shore in the water.

Your run and swim speeds are also exogenous. You might think you can control how fast you run and swim, but these are not the actual variables. As the labels show, "Max Run Speed" and "Max Swim Speed" are your fastest run and swim speeds. We are assuming you are going all out to save the victim. Sure, you could train to get faster, but at that moment when you are trying to save the victim, your max speeds are given. The fastest you can (and will) run is 5 m/sec and the fastest you can (and will) swim is 2 m/sec.

To really cement the concept of exogenous variables, ponder this: If you were waiting to order at the drive-thru at your favorite fast-food restaurant, can you think of a few exogenous variables? What makes the variables you chose exogenous?

Endogenous variables are the ones the decision-maker sets and determines. Since *endo* means inside in Greek (you have an endoskeleton), endogenous variables are within your control. They are also known as choice or dependent variables. What to drink is endogenous to you at the fast food restaurant.

You decide how far you will run from zero (dive right in and start swimming, Path B) to 100 meters (Path A) so this is the endogenous variable in this problem. Notice that deciding how far to run immediately determines how much you will swim since you will always swim in a straight line along the hypotenuse formed by the triangle of where you entered the water (see Figure 2.2). We can Use Excel to demonstrate this.

STEP Enter the formula =SQRT((A2-A8)^2+A3^2) in cell A9 and press Enter. You will know you did it correctly if Excel displays something close to 141.4217. The formula computes the hypotenuse for any value of cell A8. Excel treats the value of the blank cell A8 as zero so A2 – A8 is 100 and we get the hypotenuse of Path B (shortest total distance).

EXCEL TIP Maximize flexibility of your spreadsheets.
It would have been easier to enter and read the formula if we *hard-coded* the distance away on sand and in water like this: =SQRT((100-A8)^2+100^2), but we would lose the flexibility of being able to change these variables and have the formula automatically update. In general, you want to develop spreadsheets that depend on other cells and not on numbers.

STEP To make sure the formula works well, let's check Path A: enter 100 in cell A8. Cell A9 should also display 100. If not, return to the previous step and fix the formula.

In addition to exogenous and endogenous variables, every optimization problem has to have a *goal*, known more formally as an *objective function*. For the lifeguard problem, this is minimizing the time it takes to get to the victim. The goal always has to be a function (in Excel, a formula) that depends on the endogenous variables.

STEP Enter the formula =A8/A4+A9/A5 in cell A12 and press Enter. You will know you did it correctly if Excel displays 70 (with cell A8 at 100) since we know this is how long it takes for Path A.

Having set up the optimization problem in Excel, with exogenous and endogenous variables, along with a goal, we are now ready to find the best path. We begin by manually trying a few different paths.

STEP Set cell A8 to 50 and notice the time to victim. Try 25 and 75. Keep playing with cell A8 until you think you've found the best answer.

Cell A8 should now be in the mid 50s and the time to victim should be around 65.8 seconds. That's several seconds better than path A or B—this could mean the difference between life and death!

Not many people try fractional distances, but this is perfectly acceptable. Of course, we are not going to hunt and peck for hours, changing A8 by 0.001 to see the effect on A12. Instead, we will use Excel's Solver. Hunting and pecking (or its twin, plugging and chugging), but really fast, is exactly what Solver does.

Excel's Solver is an optimization algorithm. It tests many trial solutions very quickly, converging to an answer. Later, we will explore how it works in more detail. Right now, let's see it in action.

STEP Click the *Data* tab (in the *Ribbon* across the top of the screen), then Solver (in the Analysis group) to bring up the Solver Parameters dialog box. If Solver is not available, then use the Add-in Manager to install it. Use Excel's Help if you are having trouble or visit support.office.com.

EXCEL TIP Press *alt, t, i* to access the Add-in Manager. Press the *alt* key, then release it and press the letter *t*, then release it and press the letter *i* to quickly bring up the Add-in Manager.

The Solver Parameters dialog box is initially empty. You need to give it the appropriate information before asking it to search for a solution. It always needs specific cell addresses for the Objective and Changing Variable Cells inputs. Some problems are constrained and require further input, but the lifeguard problem does not.

STEP Do these three things: 1) Click in the Set Objective input field and select cell A12 by clicking on it. 2) Click the Min radio button. 3) Click the By Changing Variable Cells input field and select cell A8. Your screen should look like Figure 2.4. If not, make it so.

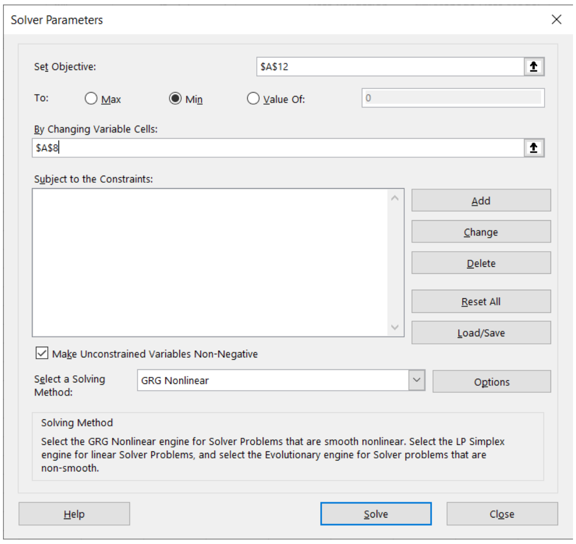

Figure 2.4: The Solver Parameters dialog box.

STEP Click the Solve button in the bottom right corner of the Solver dialog box, read the Solver Results dialog box, and click OK.

You did it! You found the correct answer to the lifeguard problem! By running about 56 meters before jumping in, you will reach the victim in a little under 66 seconds and this is the smallest possible time. We can see that this is a true minimizing solution by exploring a small move away from Solver's solution.

STEP Change cell A8 to 57 and you can see that cell A12 increases a little bit. Decrease cell A8 to 55 and, once again, cell A12 goes up. That is pretty convincing evidence that Solver has the correct answer since the values around it yield higher times to reach the drowning person.

There is more to do, but let's recap. We modeled an optimization problem, implemented it in Excel, and used Excel's Solver to find the optimal solution. There are other ways to solve optimization problems, but Solver offers a pretty simple approach—as long as you can set up the problem in Excel, Solver has a shot at solving it. We will see in future work that Solver is not perfect, but on a simple problem like this one, it is reliable.

Excel's Solver is an example of numerical methods. Its answer is not exactly correct, but it is so close that, practically speaking, it is the right answer. This problem can be solved analytically to get an exact solution and doing so yields something called Snell's Law. Physicists have known for a long time that light refracts when it goes through a different medium than air, like water. As Fermat realized in 1650, light is taking the fastest path and this is why it bends when it hits water.

There are some pretty mind-boggling applications of minimizing time across different media from the natural world. Some people think dogs can find the optimal path (Pennings, 2003 and Perruchet and Gallego, 2006) and so can ants, as shown in Figure 2.5. As described in the caption, the green area is rough felt that is harder to walk on. The ants do not go in a straight line to the food (like they do in a control setting with only the white, smooth surface). The authors call this *decentralized optimization* because the ants are working together without any direction from an authority telling them what to do.

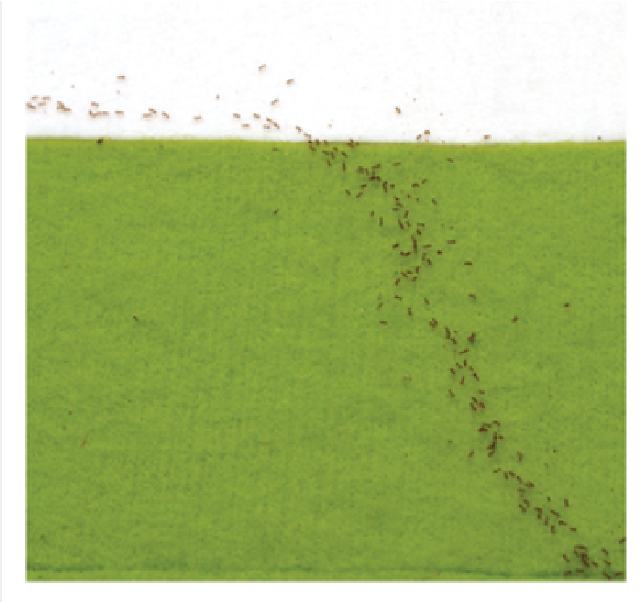

Figure 4. "Refracted" trail of *W. auropunctata* workers at the medium border between smooth (white) and rough (green) felt.
The position of the food is on the rough felt. Note that the density of workers on the rough felt is higher than on the smooth felt because travel speed is lower. In addition it appears, although not very obvious, as if the ants on the rough felt 'float' on top of the felt hairs, indicating the difficulty of walking on this substrate. Photograph kindly provided by Simon Tragust.

Figure 2.5: Ants Solving the Lifeguard Problem.
Source: Oettler, et al., 2013

Let's close with a crazy thought experiment. What would happen if you were suddenly transformed from a slow runner, near Olympic swimmer to a freakish mix of Usain Bolt and Michael Phelps? Not only are you a really fast swimmer, you also can run 100 meters in 10 seconds, so your maximum run speed is a blazing 10 m/sec. How does this shock (change in the maximum run speed exogenous variable) affect your decision of how far to run?

STEP Change cell A4 to 10 and run Solver.

Does Solver's new optimal solution make sense when compared to the initial answer?

Takeaways

We introduced and solved the lifeguard problem: to get to a drowning swimmer as quickly as possible, the lifeguard has to choose the best path.

This is a well-known problem in physics and optics because light bends (refracts) when it goes through a different medium, like when it hits water.

There are several different analytical formulas, but we used Solver (a numerical method) to find the fastest path.

We had to implement the problem in Excel, creating an objective function that depended on a changing cell. We then called Solver and it generated the correct solution.

References

The epigraph of this chapter can be found in section 26-3 of the online edition of Feynman's famous *Lectures on Physics*. He explains what Fermat did and gives the analytical solution.

For a fun visualization of the lifeguard problem, see geogebra.org/m/wBcKASpN

Oettler, J., Schmid, V., Zankl, N., Rey, O., Dress, A., and Heinze, J. (2013) "Fermat's Principle of Least Time Predicts Refraction of Ant Trails at Substrate Borders," *PlosOne*, open access: doi.org/10.1371/journal.pone.0059739

Pennings, T. (2003) "Do Dogs Know Calculus?" *The College Mathematics Journal*, 34(3) (May, 2003), pp. 178-182. jstor.org/stable/3595798

Perruchet P. and Gallego, J. (2006) "Do Dogs Know Related Rates Rather than Optimization?" *The College Mathematics Journal*, Vol. 37, No. 1 (Jan., 2006), pp. 16-19. www.jstor.org/stable/27646266