

# Parrondo's Paradox

A paradox is something (such as a situation) with opposing elements which seems impossible, but is actually true.

An optical illusion is related to a paradox in that you see something that is not easily explained or can seem impossible. Figure 3.5 is an example. Do you see the old woman or the young lady, or both? Your age affects what you see in this drawing—older people are more likely to see the old lady (Nicholls et al., 2018).



Figure 3.5: Old woman or young lady?

Juan Parrondo is a physicist who discovered the paradox named after him in 1996. Parrondo's Paradox occurs when *two losing games are combined and they produce a winning game*. That is puzzling and counterintuitive.

Almost always, game outcomes are additive, so a losing game plus a losing game equals a losing game just like adding two negative numbers gives an even more negative number. Parrondo found what can be described as a black hole in the parameter space where adding two losers yields a winner.

The paradoxical nature of the result will become apparent when we implement the games in Excel and directly examine the outcomes. Our goal is to show how simulation makes Parrondo's Paradox crystal clear.

## Losing Game A

Game A is coin flip with a slightly negatively biased coin. Heads earns you +1 monetary units (M) and tails  $-1$ . The coin is flipped 100 times and we keep a running sum after each flip. On average, at the end of the game, the result is negative so we say Game A is a losing game.

**STEP** In cell A1, enter the label "Game A." In cell A3 enter the label "epsilon" and in cell B3 enter the number 0.005. In cell A4 enter the label "p(H)," this is the probability of flipping a head. In cell B4, enter the formula:  $=0.5-B3$ .

With a probability of heads less than 50%, we will flip heads less often than tails. This is why this game is a loser.

**STEP** In cell A6 enter the label "Flip #" and create a series from 1 to 100 in cells A7:A106. In cell B6 enter the label "Result" and in cell B7, enter the formula:  $=IF(RAND()<B4,1,0)$ . Fill it down to cell B106.

Column B has our simulated coin flips. We know  $RAND()$  is uniformly distributed on the interval  $[0,1]$ . It will produce tails slightly more frequently than heads because  $B4 < 0.5$ .

We track the money in column C with an IF statement.

**STEP** In cell C6 enter the label "End M" and in cell C7 enter the formula:  $=IF(B7=1,1,-1)$ .

The formula for the next cell is different because we have to track how much money we had at the end of the previous flip. Thus, we add the cell above.

**STEP** In cell C8, enter the formula  $=IF(B8=1,1,-1)+C7$ . Fill it down to cell C106. Select C6:C106 and make a *Scatter* chart. Press F9 a few times.

The chart shows the entire game and cell C106 tells us the outcome of Game A. If it is positive, the game was won; if negative, it was lost. The number tells us how much we won or lost.

We can use the MCSim Excel add-in to examine the sampling distribution of C106. If needed, download and install MCSim from [tiny.cc/mcsim](http://tiny.cc/mcsim).

**STEP** Select cell range C7:C106 and click *MCSim* (in the *Add-ins* tab). Check the *Record All Selected Cells* option and click *Proceed*. The simulation is fast, but Excel may take some time to display the results.

Excel inserts two sheets in the workbook. The *MCSim* sheet has the result for C7 (the first coin flip), but the *MCRaw* sheet has 1,000 rows and 100 columns of numbers (which is why Excel took so long to display the results).

We can process these numbers to understand Game A. Each row is a game with 100 flips running left to right. Each flip number (column) is called an *ensemble* and the average of each flip number is called the *ensemble average*.

**STEP** In cell A1003 of the *MCRaw* sheet, enter the formula:  
 $=AVERAGE(A2:A1001)$ .

This value will agree exactly with the average in J5 of the *MCSim* sheet. This value is the average M after one flip. It is slightly negative.

**STEP** Select cell A1003 in the *MCRaw* sheet and fill it right to the 100<sup>th</sup> column (CV). With these cells selected, make a chart by clicking *Insert* and choosing *Scatter* with lines and no markers. Title the chart “Flip #” (since the title is right above the  $x$  axis it labels the axis).

Your chart is an approximation to the exact ensemble average in Figure 3.6. Simulation produces random deviation from the exact object. We could improve the approximation by increasing the number of repetitions. The squiggly graph in the simulation would converge to the line in Figure 3.6.

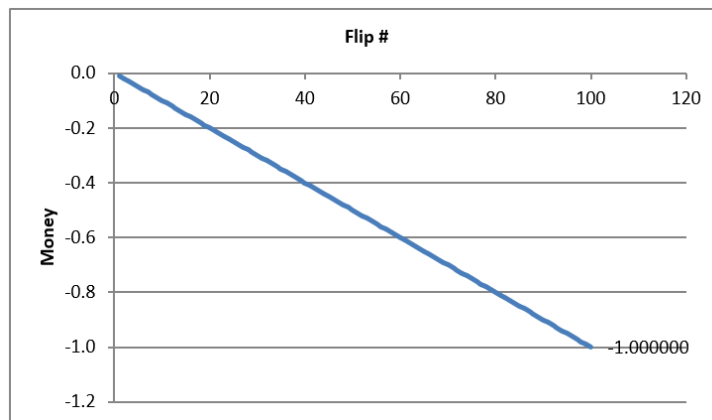


Figure 3.6: The exact evolution of Game A.

Your simulation and Figure 3.6 make clear that Game A is a loser. From the first flip, it gets steadily worse and by the last flip, you can expect to lose 1 monetary unit.

Of course, not every single game is a loser. Column *CV* in the *MCRaw* sheet shows many cells that are positive. On average, however, we can expect to lose playing Game A.

## Losing Game B

Game B is also a loser, but it is more complicated than Game A. Game B is based on two coins and you use Coin 1 if your current monetary holding is divisible by 3, otherwise you use Coin 2. The MOD function enables us to determine which coin to use.

**STEP** Return to the sheet with Game A and in cell E1, enter the formula:  $=MOD(13,12)$ .

The cell displays 1 because it is doing modulo arithmetic. Converting military time to AM/PM time uses modulo operator: 1300 is 1 PM because you divide by 12 and the remainder is the answer.

Game B follows the flow chart in Figure 3.7. We will use the MOD function to divide any number by 3 and if the result is zero, we know it is evenly divisible and we flip Coin 1. If not, we flip Coin 2.

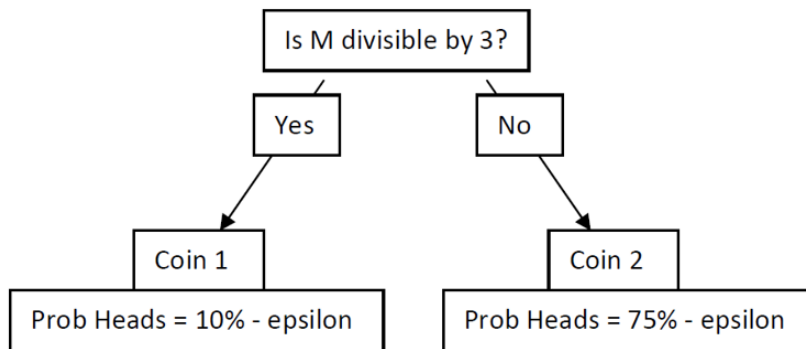


Figure 3.7: The rules of Game B.

**STEP** In cell G1 enter the label “Game B.” In cells G3 and G4 enter the labels “Coin 1” and “p(H).” In cell H4, enter the formula:  $=0.1-B3$ . In cells J3 and J4 enter the labels “Coin 2” and “p(H).” In cell K4, enter the formula:  $=0.75-B3$ .

Obviously, we would rather flip Coin 2. It comes up heads almost 75% of the time so we win 1 monetary unit. Coin 1 is the opposite. It is strongly biased against heads so we lose often with Coin 1.

**STEP** Copy cells A6:A107 and paste in cell G6. In cell H6 enter the label “Start M” and in cell H7 enter a 0. In cell I6 enter the label “MOD(M,3)” and in cell I7, enter the formula:  $=MOD(H7,3)$ . In cell J6 enter the label “Coin” and in cell J7, enter the formula:  $=IF(I7=0,1,2)$ .

You start with 0 monetary units so that is evenly divisible by 3 so we will use Coin 1, but we need a more general formula to determine what happens if it is Coin 1 or Coin 2. An IF statement can handle this.

**STEP** In cell K6 enter the label “Result” and in cell K7, enter the formula:  $=IF(J7=1,IF(RAND()<H$4,1,0),IF(RAND()<K$4,1,0))$ .

Next, we report our money position.

**STEP** In cell L6 enter the label “End M” and in cell L7, enter the formula:  $=IF(K7=1,1,-1)$ .

In the next row, we walk through the cells in order.

**STEP** In cell H8 enter the formula:  $=L7$  and fill it down. Select cell range I7:K7 and fill it down. In cell L8 enter the formula:  $=IF(K8=1,L7+1,L7-1)$ . Fill it down. Select L6:L106 and make an XYScatter chart.

This completes Game B. Cell L106 gives the final result of the game, but as we did before, we can track every flip of the game to better understand it.

**STEP** Select cell range L7:L106 and click *MCSim* (in the *Add-ins* tab). Check (if needed) *Record All Selected Cells* and click *Proceed*.

As before, two sheets are inserted and we will process the data in the *MCRaw* sheet to show how Game B works.

**STEP** Return to the *MCRaw3* sheet and copy row 1003. Go to the *MCRaw5* sheet, select cell A1003 and paste. Make a chart of row 1003.

Your results are surprising. In the first few flips of the ensemble average, it jaggedly oscillates and then settles down to a downward sloping relationship.

The exact ensemble average is given by Figure 3.8. The simulation is correct in that there is an oscillation in the expected value in the first few flips before convergence to a single line that heads downward.

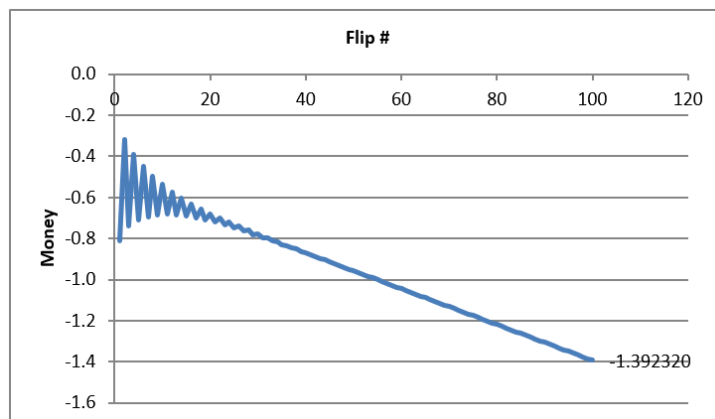


Figure 3.8: The exact evolution of Game B.

Your simulation results and Figure 3.8 show that Game B is a loser and a bigger loser than Game A. We can expect to lose about 1.4 monetary units playing Game B.

## Mixing Two Losing Games

Having set up and run games A and B separately, we are now ready to mix these two losing games. This will demonstrate Parrondo's Paradox because, somehow, mixing the losing games results in a winning game. In fact, there is an optimal mixing strategy, but we will randomly mix the two games by flipping a fair coin.

**STEP** In cell N1 enter the label "Random Mixing." Copy cell range A6:A106, select cell N6, and paste. In cell O6 enter the label "Game" and in cell O7, enter the formula:  $=IF(RAND()<0.5, "A", "B")$ . Fill it down.

Column O tells us which game we will play at each coin flip. It is easy to see by pressing F9 repeatedly that the letters A and B are bouncing around indicating that we are mixing the games randomly.

We need to input Game B again (we cannot just use the Game B in columns G:L) because it depends on value of M to decide which coin to play.

**STEP** In cell P5 enter the label "If Game B is chosen." Copy cell range H6:K7, select cell P6, and paste.

We need an IF statement to display the actual outcome of this game based on whether we play Game A or Game B. We take Game A from column B since it does not depend on the amount of M we have, but we take Game B from column S.

**STEP** In cell T6 enter the label "Actual Result" and in cell T7 enter the formula:  $=IF(O7="A", B7, S7)$ .

Next, we determine our monetary position.

**STEP** In cell U6 enter the label "End M" and in cell U7, enter the formula:  $=IF(T7=1, 1, -1)$ .

We process the second flip and fill down to complete the implementation.

**STEP** In cell P8 enter the formula  $=U7$ . Fill it down. Select cell range Q8:T8 and fill it down. In cell U8 enter the formula  $=IF(T8=1, U7+1, U7-1)$ . Fill it down. Select cell range U6:U106 and make a *Scatter* chart.

This completes the random mixing of two losing games. Cell U106 gives the final result of the game. Pressing F9 does not reveal much. We need to run a simulation.

**STEP** Select cell range U7:U106 and click *MCSim* (in the *Add-ins* tab). Confirm that the *Record All Selected Cells* option is still checked and click *Proceed*.

We have the data to demonstrate Parrondo's Paradox, but we need to create an ensemble average chart.

**STEP** Return to the *MCRaw5* sheet and copy row 1003. Go to the *MCRaw7* sheet, select cell A1003 and paste. Make a chart of row 1003.

The results are absolutely stunning. Unlike our two previous charts, this one points upward and the final value is positive! This is a winning game! Figure 3.9 shows the exact evolution of the randomly mixed games. The expected value of playing a random combination of games A and B keeps rising the more you play. That is mind-boggling.

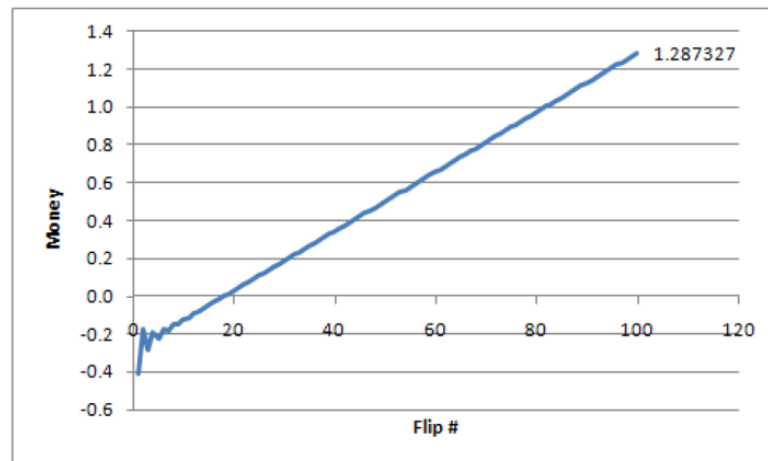


Figure 3.9: The exact evolution of randomly mixing A and B.

By randomly mixing individually-losing Games A and B, we can expect to win about 1.3 monetary units playing 100 times. This is Parrondo's Paradox.



## What is Going on Here?

Does this work show that you can walk into a casino and take turns playing blackjack and roulette and come out a winner? No.

Does it mean that you can combine two losing stocks and somehow make money? No.

Does it mean that I can take something poisonous and then drink another poison and the two will combine to heal me? No.

Parrondo's Paradox does not say that mixing *any* two losing games produces a winner. The two games and epsilon value were chosen carefully. Parrondo found a parameter value that generated the anomalous result. Think of a Cartesian plane with a coordinate that is like a black hole—all of the other coordinates behave as expected, but this particular point is really weird. Parrondo found such a point by carefully picking the bias (epsilon) in games A and B.

Applying this paradox to the real-world is a challenge. Explaining the inspiration for Parrondo's discovery of the paradox will help us understand how the paradox emerges.

Parrondo is a physicist and his discovery of an epsilon value that produced the paradox was influenced by something called the *flashing Brownian ratchet*. This is a process that alternates between two regimes in a sawtooth fashion.

**STEP** Watch this 2-minute video to see the ratchet in action and how it applies to Parrondo's Paradox: [vimeo.com/econexcel/parrondo](https://vimeo.com/econexcel/parrondo). You can control the ratchet yourself here: [tiny.cc/hbratchet](https://tiny.cc/hbratchet).

It is true that Parrondo's Paradox requires a specific, and we might add rare, type of losing game to be mixed. The paradox would never emerge if the two losing games were like Game A. Mixing two Game As would produce a bigger negative outcome.

Game B, with its two coins, one of which is biased in our favor, holds the key to the paradox. Figure 3.8 tells us that, for the first few flips, the expected value of Game B alternates. Mixing takes advantage of the positive parts of Game B in those first few flips.

## Takeaways

Optical illusions and paradoxes are mind-bending. They violate what we expect to happen and force us to deal with something unbelievable.

*STEP* Watch a classic 2-minute video of an optical illusion with an explanation of how it works: [tiny.cc/hbfaceillusion](http://tiny.cc/hbfaceillusion).

Like an optical illusion, Parrondo's Paradox produces a shocking result: loser plus loser equals winner. That should not happen.

At a magic show, we know there is a trick involved, so the person was not really cut in half or made to disappear. Parrondo's Paradox is explained by the ratchet, which catches the ball at just the right time and pushes it upwards, producing an overall upward movement.

Parrondo's Paradox requires specific values for epsilon, the bias in coins being flipped, and Game B is actually a combination of two coins which are used based on whether the player's total amount of money is evenly divisible by three.

We used simulation to explain Parrondo's Paradox, but it is not an integral part of the paradox. An analytical solution using Markov chains provide exact results. The analytical solution was used to create the exact evolution charts (Barreto, 2009).

Usually, we use simulation to represent a real-world process. We do not have to make it a perfect representation, but it must capture the essential elements.

We can go, however, in the other direction, from an artificial environment to the real world. Parrondo found something paradoxical and now we are asking, "Is there something like this in reality?" Could it ever make sense to combine stocks or medicines or anything else in a way that reverses the negative result? The search is on.

Finally, random mixing produces a winning game with an expected value of about 1.3 monetary units at the 100<sup>th</sup> flip, but there is an optimal mix. Playing AB, then ABBABBABB repeatedly, produces an expected value of a little over 6 monetary units. Barreto (2009) explains the analytical solution and optimal mixing with Excel.

## References

Barreto, H. (2009) “A Microsoft Excel Version of Parrondo’s Paradox” *SSRN Working Paper*:

[academic.depauw.edu/~hbarreto/working/Parrondo/ParrondoSSRN2.pdf](http://academic.depauw.edu/~hbarreto/working/Parrondo/ParrondoSSRN2.pdf) and  
[academic.depauw.edu/~hbarreto/working/Parrondo/Parrondo.xls](http://academic.depauw.edu/~hbarreto/working/Parrondo/Parrondo.xls).

Nicholls, E., Churches, O., and Loetscher, T. (2018) “Perception of an ambiguous figure is affected by own-age social biases” *Scientific Reports* volume 8, Article number: 12661. Open-access [www.nature.com/articles/s41598-018-31129-7](http://www.nature.com/articles/s41598-018-31129-7).

A Mathematica version:

[demonstrations.wolfram.com/TheParrondoParadox/](http://demonstrations.wolfram.com/TheParrondoParadox/)

A YouTube demonstration:

[www.youtube.com/watch?v=PpvboBJEozM](http://www.youtube.com/watch?v=PpvboBJEozM)

For an entertaining read on paradoxes, try *Perplexing Paradoxes* by George Szpiro (2024).