The most powerful force in the universe is compound interest.

attributed to Albert Einstein

Growth Math

The learning objective is straightforward to state, but difficult to accomplish: a deep appreciation of the power of compounding. You want to go beyond the simple mechanics of growth and truly grasp the nature of the force unleashed by exponential growth.

In addition, there are two subgoals:

1. Measuring growth via the CAGR, the compound annual growth rate.

2. Understanding and using the Rule of 70.

A Race

We are going to pit an arithmetic against a geometric sequence (or progression). This is not a mystery so we will reveal right now that the geometric sequence will win. Be on the lookout, however, for some surprising results.

An *arithmetic sequence* is a list of numbers with a common difference between each term. The sequence 4, 9, 14, 19, 24 is an arithmetic sequence with 5 as the difference.

Instead of a constant additive increase, a *geometric sequence* progresses by applying a multiplicative constant to each term. So, 4, 8, 16, 32 is a geometric sequence with 2 as the multiplier. Notice that doubling is a 100% increase.

Starting from 4, it is easy to see that doubling beats adding 5 pretty quickly—by the third term. But what if we made it really uneven in choosing the additive and multiplicative constants?

Starting with \$1, what if you got \$1,000,000 more every day versus a 10% per day increase? We will use Excel to run this race.

Let's agree that the arithmetic sequence is going to jump out to a big lead. After two days, it has \$2,000,001 while the geometric sequence will have \$1.21. But what happens as time goes by?

STEP Save a blank Excel workbook as *Growth.xlsx* and enter the labels in row 2 as shown in Figure 4.1. In cells B1 and C1, enter the values \$1000000 and 10% (including the \$ and %) and name the cells x and i (for interest rate). Use Excel's Help or the web if needed to name the cells and widen column B until the value is visible.

> *EXCEL TIP* Naming cells improves presentation and makes your implementation easier to follow. Using natural language text in formulas instead of cell addresses requires more setup time, but the improvement in readability of formulas is well worth the effort.

STEP Enter 0 and 1 in cells A3 and A4, respectively, then select both cells, click the bottom-right corner of cell A4, and drag down to cell A50. In cell B3, enter \$1 and in cell B4 enter the formula *=B3+x*. Select cell B4 and double-click at the bottom-right corner to fill it down. Enter \$1 in cell C3 and the formula *=(1+i)*C3* in cell C4. Fill it down.

	A	B	C
1		\$ 1,000,000	10%
2	Day	Amount	Amount
3	0	\$ 1	\$ 1
4	1	\$ 1,000,001	\$ 1.10
5	2	\$ 2,000,001	\$ 1.21
6	3	\$ 3,000,001	\$ 1.33
7	4	\$ 4,000,001	\$ 1.46

Figure 4.1: Setting Up the Race.

The formula in column C is produced by this algebraic simplification of the way the next term is produced in a geometric sequence:

$$x_{t+1} = x_t + ix_t$$
$$x_{t+1} = (1 + i)x_t$$

STEP Widen columns B and C to make sure the numbers are visible in row 50. It is obvious that $1M per day is way ahead of 10% per day, but let's make a chart to show how far ahead the arithmetic sequence is—select cell range A2:C50 and insert a *Scatter* chart. Give it a title (you can use *Racing Sequences*), label the x axis (*Day* would work), and insert text boxes without fills or outlines to label the two series.

Your chart has a line with a slope of $1M and what appears to be another line (the geometric sequence) on the x axis. The values of the geometric sequence are so small, you cannot tell that it is actually a curve. What happens if we extend the sequence?

STEP Select cells A50:C50, click the bottom-right corner of cell C50 and drag down to row 153. Widen column C to see the values and decrease the decimal places so only integer dollar values are displayed to make it easier to compare columns B and C.

After 150 days, the arithmetic sequence is still way ahead,roughly $150M to $1.6M, but we can see that the geometric sequence is starting to really gather momentum.

STEP Extend the sequences to row 253.

After 250 days, the geometric sequence is almost 100 times bigger—almost $25 billion compared to $250 million. When did the geometric pass the arithmetic sequence?

STEP Scroll back up to the top of the sheet, enter the label *Difference* in cell D2 and the formula *= C3 - B3* in cell D3. Double-click the bottom-right corner of cell D3 to fill it down. Scroll down and widen column D as needed as you scroll.

You will see that the parentheses (indicating negative numbers) stop on day 201. That is the day the geometric sequence won the race and its lead will grow wider, ever faster, from then on.

STEP Scroll back up and edit the SERIES formulas in the chart. Change the row numbers to 213 so that the formula for SERIES 1 looks like this:
=SERIES(Sheet1!$B$2,Sheet1!$A$3:$A$213,Sheet1!$B$3:$B$213,1)
Do the same for the second series.

Figure 4.2 shows what your chart should look like. Do not be misled into thinking that the geometric sequence was not growing at first and then started growing really fast around day 150. In fact, it grew at the same rate, 10%, every single day. Another mistake is to see every curve as having constant growth—do not fall into this trap.

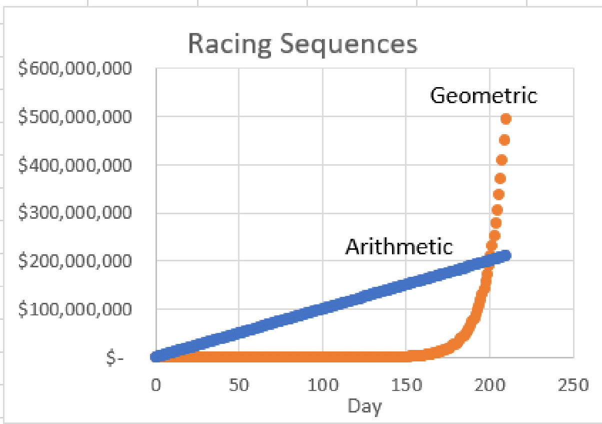


Figure 4.2: The Geometric Sequence Wins!

One quick way to check if a curve is growing at a constant rate is to make the y axis a log scale.

STEP Click on the y axis (the $ values) and check the *Logarithmic scale* box in the *Axis Options* on the right of your screen.

The chart dramatically changes. The curve becomes a line and the line a curve. The fact that a log scale linearizes the curve means the curve is growing at a constant rate.

Now, you might think that we are at the surprising result mentioned at the beginning. After all, it is pretty impressive that 10% per day, after starting so incredibly far behind and falling even farther behind, overtakes $1M per day, but no, that is not it.

The big surprise is actually that it does not matter what (positive) numbers you pick for the constant difference and multiplicative factor, the geometric will ***always*** eventually beat the arithmetic sequence.

Let's be clear about this. You can make the constant difference as big as you want and the multiplicative factor as little as you want (as long as it is positive) and the geometric progression will eventually win the race. That is shocking and reveals the force embedded in compounding.

You could argue that this is expected because multiplication is more powerful than addition and that is a true statement, but Figure 4.2 hints at another way to remember why geometric progressions are so powerful—they are curves instead of lines. Eventually, if they start from the same point and are both increasing, a curve will always pass a line.

You have undoubtedly heard about the power of compounding and it is true that compounding is an incredibly important concept in business. You want, however, to have a deep appreciation of the idea that compounding over long periods of time will produce remarkable results.

STEP Just to be sure and to give you another wow moment, reduce the multiplicative constant in cell C1 to 1%. Will growing at 1% per day catch and beat $1,000,000 per day? Amazingly, yes, you know it will, but when? Find the day the geometric sequence beats the arithmetic one. The answer is in the appendix.

CAGR

We can measure the rate of growth between any two points by using a formula called the *compound annual growth rate*, CAGR. The word annual is used because it is often applied to yearly data, but we can apply the CAGR to the daily frequency in the race we just ran. Instead of just stating the formula, it is worth seeing where it comes from.

We know that a geometric sequence is generated by adding a constant multiplicative factor (i) times the previous amount. Here are the first few terms, where x_0 is the initial value, x_1 is the next value, and so on.

$$x_1 = x_0 + ix_0 = (1+i)x_0$$
$$x_2 = x_1 + ix_1 = (1+i)x_1$$
$$x_3 = x_2 + ix_2 = (1+i)x_2$$

The subscript tells us the time period, with zero meaning right now. The last equation above says the value of x in time period 3 equals the previous value of x plus the rate of growth times the previous value. The value of x in time period 3 also can be expressed as $(1+i)$ times the value in time period 2.

We can rewrite x_2 by substituting in the formula for x_1 like this:

$$x_2 = (1+i)x_1 = (1+i)(1+i)x_0 = (1+i)^2 x_0$$

We can do the same for x_3:

$$x_3 = (1+i)x_2 = (1+i)(1+i)(1+i)x_0 = (1+i)^3 x_0$$

In fact, we could do this for any value in the sequence after the initial value:

$$x_t = (1+i)x_{t-1} = (1+i)^t x_0$$

The equation above says that starting from an initial value, x_0, the value of x at time t, x_t, will be $(1+i)^t$ times the initial value. We do not have to know the previous value to get the next value. All we need is the initial value, the rate of growth, i, and the number of time periods.

Since this equation expresses where every point will be in the sequence, we can solve for i like this:

$$x_t = (1+i)^t x_0$$
$$\frac{x_t}{x_0} = (1+i)^t$$
$$[\frac{x_t}{x_0}]^{1/t} = 1+i$$
$$[\frac{x_t}{x_0}]^{1/t} - 1 = i$$
$$i = [\frac{x_t}{x_0}]^{1/t} - 1$$

The last equation is the CAGR. We can write it in a more user-friendly way:

$$i = [\frac{x_t}{x_0}]^{1/t} - 1$$
$$CAGR = [\frac{\text{Final Value}}{\text{Initial Value}}]^{1/\text{Number of Time Periods}} - 1$$

Given any two numbers and the number of time periods, the CAGR tells us what the rate of growth must be if the numbers are part of a geometric sequence. Let's put this formula to work.

STEP Label cell E1 as *CAGR* and enter the formula *=(C9/C3)ˆ(1/6)-1* in cell E2.

You calculated the rate of growth from the initial value of \$1 at $t = 0$ to the value in the 6th time period and this will equal the rate of growth in cell C1.

Be careful with the number of time periods in the CAGR formula. From $t = 0$ to $t = 6$ and from cells C3 to C9 there are seven numbers, not six. The number of periods is always one less than the number of values in the sequence. The number of time periods, t, is the amount of time that has elapsed since the start. For a length of one unit of time, you need two numbers, beginning and end.

STEP Modify your formula in cell E2 to compute the CAGR from time period 5 to 10. You will know you have it right if cell E2 equals cell C1.

Of course, we constructed the geometric sequence in column C so we know the rate of growth. What if we did not know the rate of growth and had only beginning and ending values?

STEP Click on an empty cell and compute the CAGR from an initial value of 11.7 in $t = 0$ to 23.1652 in $t = 14$.

The correct formula is =(23.1652/11.7)ˆ(1/14)-1 and Excel should be displaying 0.05. Notice that we do not need to know the numbers in the intervening time periods. If it is a geometric sequence, that is, growing at a constant rate, then we could compute the value at any time period using the generating equation, $x_t = (1 + i)^t x_0$.

Let's show that values between initial and final are irrelevant and introduce another measure of growth, *the average annual percentage change* (AAPC).

STEP Insert a new sheet and enter the formula *=RAND()*1* in cell A1, *=RAND()*2* in cell A2, and =RAND() times 3, 4, and 5 in cells A3, A4, and A5. Make a graph by selecting range A1:A5 and inserting a *Scatter* chart. Press F9 (you may have to use the *fn* key on your keyboard) a few times to recalculate the sheet.

Even though the values do not grow at a constant rate, we can compute the CAGR from A1 to A5.

STEP Label cell A6 *CAGR* and, in cell A7, enter the formula to compute it from A1 to A5.

The formula you entered (to be sure: *=(A5/A1)^(1/4)-1*) ignores the three points in between the first and last points. The CAGR assumes that a constant growth curve connects the first and last points.

There is another common measure of growth that does use all of the points, the average annual percent change.

STEP In cell B2, enter the formula *=(A2-A1)/A1* and fill it down.

Column B has the percentage changes from one year to the next. The average annual percent change takes the average of the percentage changes.

STEP Label cell B6 *AAPC* and enter the formula *=AVERAGE(B2:B5)* in cell B7. Press F9 a few times.

It is clear that the two measures are different. The CAGR answers a specific question: what is the constant rate of growth that would need to be applied to the initial value so that we end up at the final value? The AAPC does not have the property that applying the growth rate to the first value produces the last value. The AAPC is just an average of the annual percentage changes.

In fact, there are many more ways to measure growth than CAGR and AAPC, but these are the two most common ones. And, there are many more averages than the usual one. There is one called the *geometric mean* (mean and average are synonyms). Instead of adding the values and dividing by n (the number of values), you multiply them and then take the $\frac{1}{n}^{\text{th}}$ root.

STEP Enter the formula *=A2/A1* in cell E2. Fill it down to E5. In cell E6, enter the formula =(E2*E3*E4*E5)^(1/4).

This is the geometric mean of the ratios in cells E2:E5. There is an easier way: use Excel's GEOMEAN function.

STEP In cell E7, enter the formula *=GEOMEAN(E2:E5)*. Confirm cells E6 and E7 are equal.

You probably have not noticed, but there is an important discovery at hand.

STEP In cell E8, subtract 1 from the geometric mean in cell E6 or E7 (since they are the same).

Do you see it? Look carefully at cells A7 and E8—the CAGR and geometric mean of the ratios minus 1 are the same! That is striking.

The geometric mean has applications when the data generated come from a geometric sequence. For example, if an investment is growing at a constant percentage, we might use the geometric mean because, like the CAGR, it has the property that the growth rate applied to the initial value will equal the final value.

The Rule of 70

The growth rate of a geometric sequence can be used to determine the time needed to double. We use the generating equation, but this time we know we want the initial value to double and we want to solve for t:

$$\begin{aligned} x_t &= (1+i)^t x_0 \\ 2x_0 &= (1+i)^t x_0 \\ 2 &= (1+i)^t \end{aligned}$$

With t as an exponent, we face a challenge in solving for t. The path forward involves the logarithm, which is the inverse of exponentiation. We can take the natural log, *ln*, of both sides to solve for t:

$$2 = (1+i)^t$$
$$ln(2) = tln(1+i)$$
$$\frac{ln(2)}{ln(1+i)} = t$$
$$t = \frac{ln(2)}{ln(1+i)}$$

We can use this formula to find the exact time it will take a geometric sequence to double. If the growth rate is 10% per time period, we plug that into the formula and compute it.

STEP Return to the sheet where you raced the sequences and set cell C1 to 10%. In cell G1, enter the label *Exact Time to Double*. In cell G2, enter the formula *=LN(2)/LN(1+i)*.

You used Excel's natural log function, LN(), to compute that it will take a little over 7.2725 time periods for a geometric sequence growing at 10% to double.

Since the exact answer cannot be easily computed, an approximation is often used. It relies on the fact that $ln(1+x) \approx x$.

STEP In cell G3, enter the formula *=LN(1+i)*.

With $i = 10\%$, $ln(1+i)$ is almost 0.1, confirming the fact. In addition, ln(2) is roughly 0.693, or rounded to two decimal places, 0.70. This means we can approximate the exact answer like this:

$$t = \frac{ln(2)}{ln(1+i)}$$
$$t \approx \frac{0.7}{i}$$

We have derived the *the Rule of 70*, an approximation which can we write in a more friendly way like this:

$$\text{Time to Double} = \frac{70}{\text{Growth Rate (in percentage)}}$$

If the growth rate is 10% per day, the Rule of 70 says it will take 70 divided by 10 or 7 time periods to double. This is not exactly true. The exact time to double is displayed by cell G2, but it is reasonably close.

STEP With cell C1 at 10%, notice that the initial value of \$1 almost doubles by the 7th day and almost doubles again (to \$4) by the 14th day.

We can try a different growth rate to see if the Rule of 70 works again. At 2% per day, the Rule of 70 says it will take about 35 days to double. Is this true?

STEP Change cell C1 to 2%. Did the Rule of 70 work?

Yes, cell G2 shows it will take just a little longer than 35 days to double at 2%. Also, the value in column C is \$2 at t=35 and if you scroll down, you will see that it is \$4 at t=70. It really does double almost every 35 days with a 2% per day growth rate.

STEP Set cell C1 70% and look at the values in column C.

The Rule of 70 is not doing well here. It is not true that we get doubling every year. This shows that the Rule of 70 works for "small" growth rates and the smaller, the better the approximation.

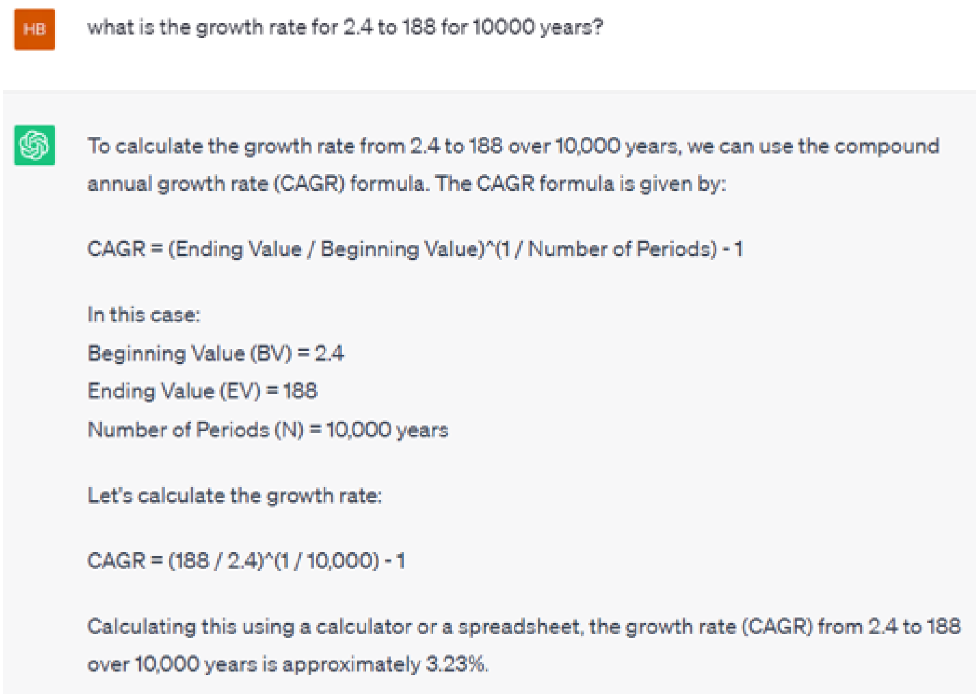
There is also the Rule of 72 which is a slightly worse approximation to $ln(2) \approx 0.693$, but has more divisors than the Rule of 70.

Growth Numeracy

Today's world population is roughly 8 billion, but humans started with small numbers and expanded. Galor (2022, p. 46) says,

> At the eve of the Neolithic Revolution [the birth of agriculture], in the year 10,000 BCE, an estimated 2.4 million human beings roamed the earth. Yet, by the year 1 CE, as the Roman Empire and the Mayan civilization approached their height, the world's population had multiplied seventy-eight fold and soared to 188 million.

Generative AI exploded in the public's consciousness in November 2022 when OpenAI released its latest version of ChatGPT. This was followed by many competitors. As part of my exploration and interaction with these new tools, I repeatedly tried to get various AIs to compute the CAGR for the increase in human population from 2.4 to 188 million over 10,000 years. Figure 4.3 shows one answer from ChatGPT 3.5 (free version) on June 7, 2023.



HB what is the growth rate for 2.4 to 188 for 10000 years?

To calculate the growth rate from 2.4 to 188 over 10,000 years, we can use the compound annual growth rate (CAGR) formula. The CAGR formula is given by:

CAGR = (Ending Value / Beginning Value)^(1 / Number of Periods) - 1

In this case:
Beginning Value (BV) = 2.4
Ending Value (EV) = 188
Number of Periods (N) = 10,000 years

Let's calculate the growth rate:

CAGR = (188 / 2.4)^(1 / 10,000) - 1

Calculating this using a calculator or a spreadsheet, the growth rate (CAGR) from 2.4 to 188 over 10,000 years is approximately 3.23%.

Figure 4.3: ChatGPT 3.5 computing CAGR.

The AIs always do a good job of explaining how to do the problem, but sometimes give ridiculous answers. For example, the steps in Figure 4.3 are correct, but the final answer is nonsensical. At 3% per year, the Rule of 70 says there will be doubling around every 25 years. That would be 16 times in 100 years. We pass 78-fold before 200 years.

A moment's reflection shows that a 3% growth rate over such a long period of time is going to produce a huge number. How huge? Excel says 1.03 to the 10000th power is 2×10^{128}. Today's world population of 8 billion is 8×10^{9} so ChatGPT's answer is not in the ballpark.

STEP Use Excel to compute the CAGR for Galor's numbers: initial value of 2.4 and final value of 188 over 10,000 years. You can use ChatGPT's CAGR formula in Figure 4.3 since it did get that right.

In fact, the CAGR is tiny, about 0.000436. Rounding roughly to 0.0005, that is 0.05% and the Rule of 70 would give doubling every 1,400 years. Now that CAGR makes sense.

Lesson: Never trust generative AI with a mathematical computation. More broadly, never trust any fact produced by an AI. Always check its claims.

ChatGPT 4 (the paid version in 2023) with a Mathematica plugin gets the CAGR computation right. But I still check every number it produces. AI will undoubtedly get better, but I will remain skeptical of any factual claim it makes. You should also.

As a final example, Poundstone (2016, p. 211) asked this survey question: Suppose you put $1,000 in a tax-free account that earns 7 percent per year on this investment. How many years will it take to double your original investment, to $2,000?

1. Between 0 and 5 years
2. Between 5 and 15 years
3. Between 15 and 45 years
4. More than 45 years

Only 59% got it right. The Rule of 70 gives 10 years so the correct answer is between 5 and 15 years. It cannot be 0 to 5 since 7% of $1,000 is $70. So, $1,070 next year, $1,147 year 2, and there's no way it reaches $2,000 in five years. Likewise, 15 to 45 and more than 45 are obviously wrong since $70 per year (ignoring compounding) times 30 years is over $2,000.

More importantly, those getting the correct answer "reported $32,000 more personal annual income, more than twice as much in savings, and rated themselves 15 percent happier" (Poundstone, 2016, p. 212). Maybe being numerate has its advantages.

Takeaways

In everyday English, exponential means really fast. In math, it means there is an exponent involved. Geometric sequences are exponential because they can be written with a generating equation like this: $x_t = (1 + i)^t x_0$.

Geometric sequences are much more powerful than arithmetic sequences, especially over a long time period. It is hard to believe, but true that a geometric will always surpass an arithmetic sequence, no matter the positive constants used.

Mathematicians usually stress the multiplicative nature of geometric sequences to explain their power, but it is also true that, starting from the same place and pointing up, a curve will always eventually pass a line.

We compute the compound annual growth rate with this formula:

$$CAGR = [\frac{\text{Final Value}}{\text{Initial Value}}]^{1/\text{Number of Time Periods}} - 1$$

The geometric mean (Excel function GEOMEAN()) is another way to compute the CAGR.

The Rule of 70 is mental math. You can quickly roughly compute how long it will take to double by dividing 70 by the growth rate. You can use the Rule of 70 to check a computed growth rate for reasonableness.

The mathematics of how things grow, CAGR, geometric mean, and the Rule of 70 are all part of being numerate. We apply these ideas to economic growth in the next section.

References

Searching the web reveals that it is pretty clear that he never said it, but the quote in the epigraph attributed to Albert Einstein certainly rings true. He is also supposed to have said something like, “Compound interest is the Eighth Wonder of the World,” but this is also doubtful. A good one that the person actually said is Charlie Munger (Warren Buffett’s partner): “The first rule of compounding is to never interrupt it unnecessarily.”

Galor, O. (2022) *The Journey of Humanity* (Dutton).

Poundstone, W. (2016) *Head in the Cloud* (Little Brown and Company), archive.org/details/headincloudwhykn0000poun

Appendix

A geometric sequence with a growth rate of 1% per day will pass an arithmetic sequence with a constant difference of $1M on day 2,161. Yes, that is roughly 10 times longer than it takes the geometric progression growing at 10% per day. And, yes, if you tried 0.1%, it would take 10 times longer. But no matter how small you make the growth rate or how big you make the constant difference, eventually, the geometric sequence wins!