Constrained Optimization

Your doctor says that you are killing yourself by smoking too many cigars and drinking too much brandy. She restricts your consumption to a total of 5.

You can have 5 brandies and no cigars or 2, 3 or any other combination that adds up to 5 or fewer. You can choose fractional amounts like one-third brandies and four and two-thirds cigars or 1.5 and 3.5.

You cannot, however, maintain your current daily consumption of 3 and 10. That adds up to 13 and violates the constraint.

Your utility function remains the same:

$$U = 18B - 3B^2 + 20C - C^2$$

But we need to add the constraint to properly state this optimization problem. Mathematically, the doctor said that $B + C \leq 5$. Formally, we write the problem like this, where s.t. stands for subject to:

$$\max_{B,C} U = 18B - 3B^2 + 20C - C^2$$
$$\text{s.t. } B + C \leq 5$$

You wonder how you are going to solve this problem. It can be done analytically, but you ask yourself if Solver can do it? Yes, of course it can.

As usual, we will have to implement the problem in Excel. Then we can run Solver to find the optimal solution.

Implementing the Problem in Excel

We have the unconstrained version of the problem already set up so we start from there.

STEP Make a copy of the sheet in your *UtilityMax.xlsx* workbook by right-clicking the *Sheet1* sheet tab in the bottom left, selecting *Move or Copy...*, checking *Create a copy*, and clicking *OK*. Rename the sheet *conopt*.

The new information that we have to implement is the constraint. It is convenient to rewrite the constraint as $B + C - 5 \leq 0$. This way, violations of the constraint occur whenever $B + C - 5$ is positive.

STEP In cell A6, enter the formula *=A1+A2-5*. In cell B6, enter the label *constraint*.

Now we are ready to call Solver. Excel will include the choices you entered before in the Solver dialog box, but we have to add the constraint.

STEP In the *Data* tab, click *Solver*. Confirm that the objective function, max, and changing cells are correct, then click the Add button. From the Add Constraint dialog box, click on cell A6 for the *Cell Reference* field, choose *le*, and enter a 0 in the *Constraint* field. Click *OK*. Notice that Excel adds the constraint to the Solver dialog box. Click *Solve*. Excel announces success! Click *OK*.

We interpret Solver's answer as $B^* = 1$ and $C^* = 4$ and $U^* = 79$, which is the correct answer. To maximize utility subject to the constraint of no more than 5 combined units of cigars and brandies, the best combination is 1 brandy and 4 cigars. This yields a utility of 79 and cannot be beat by any other combination of brandies and cigars that does not violate the constraint.

Notice that the constraint cell is not exactly zero, but it is very close to zero. Cell A6 shows something like 5.02825E-08. This is a number expressed in scientific notation and it means 5.02825 x 10^{-8}. This is a tiny number. You can make it exactly zero if you wish, by changing cells A1 and A2 to 1 and 4, respectively.

Visualizing the Optimal Solution

Figure 6.6 displays a 3D surface and contour plot of the constrained utility maximization problem. On the 3D surface plot, the constraint is a barrier that blocks attainment of the unconstrained max at the top of the hill. On the contour plot, the constraint is a line because if you look straight down at the barrier from directly above, you would see just a line.

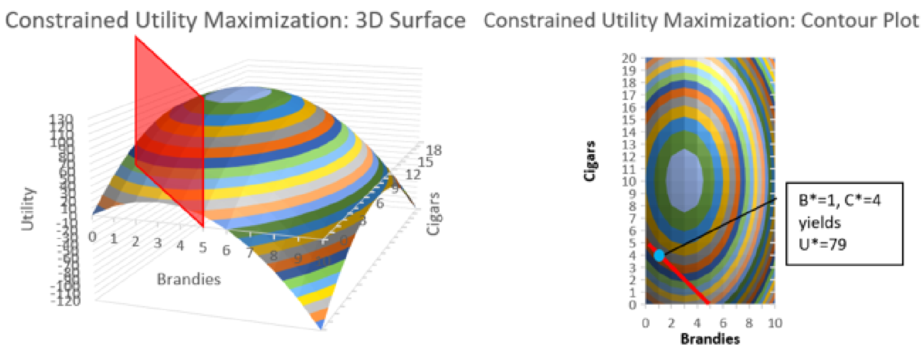

Figure 6.6: Visualizing constrained utility maximization.

STEP Make two separate lists of things you like and do not like for the 3D surface plot and the contour plot. See the appendix if you need help, but do not immediately go there. Try to make the lists yourself before looking at the appendix.

One thing that is extremely important about the contour plot is that it clearly reveals the optimal solution. We can use Excel's wire-frame version of the contour plot to see this.

STEP Copy and paste your contour plot, then click *Change Chart Type* in the *Design* tab. Select the *Wireframe Contour* in the *Surface* group of charts.

It is now much easier to see the contour lines. Since there is a contour line for every value of utility, there are actually an infinity of contour lines. Your chart shows only a few of them.

STEP Carefully place a line (use the *Line* object in the *Shapes* collection) from the point 0, 5 to 5, 0 as shown in Figure 6.7.

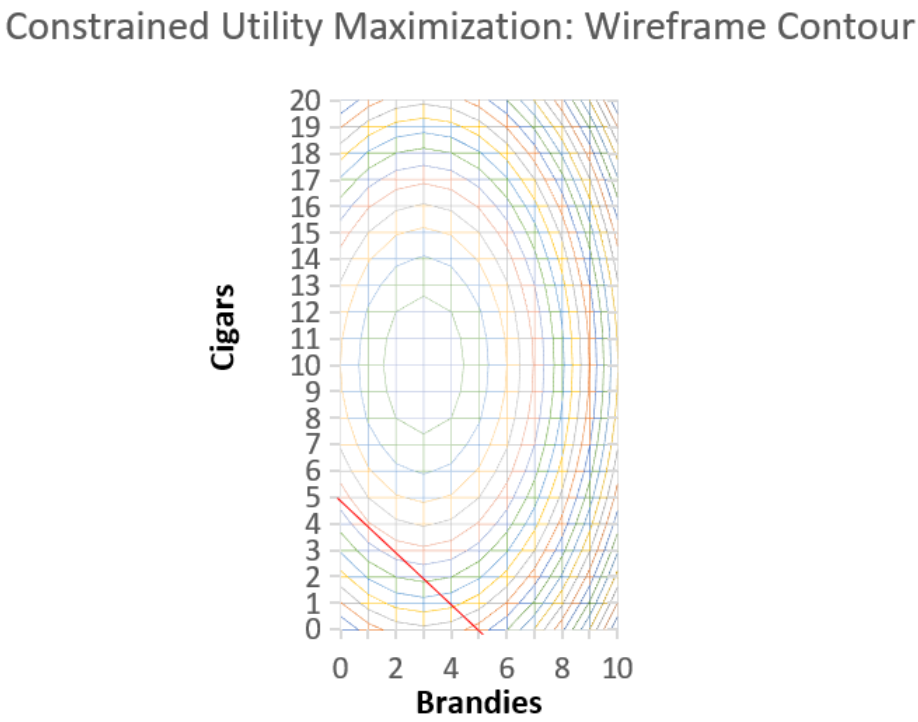

Figure 6.7: Understanding contour lines with *Wireframe* chart.

At the optimal solution (at 1, 4), there is a contour slightly above and another below. The curve above has a utility value greater than 79 and the one below is less than 79. There is a curve not shown in Figure 6.7 that is in between the two contours above and below point 1, 4. This contour just touches the constraint line and it has a utility value of exactly 79. This contour is *tangent* to the diagonal constraint.

A point of tangency means that there is contact at a single point, but no crossing. Figure 6.8 shows why tangency is a visualization of the optimal solution.

If the constraint line cuts or intersects a curve, we immediately know this is not the optimal solution. If we are cutting a contour, we can move along the constraint line to reach a higher contour.

In Figure 6.8, the highest curve ($U > 79$) is beyond our reach. The one that just touches the line is the best we can do. Tangency instantly reveals the solution and explains why the 2D contour plot is used to visualize constrained optimization.

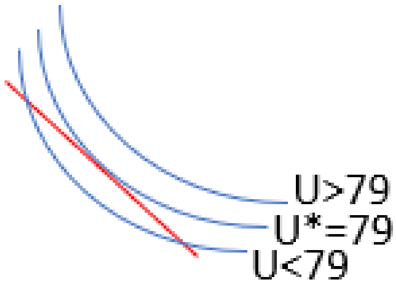

Figure 6.8: Tangency displays the optimal solution.

Takeaways

Excel's Solver add-in can handle constrained optimization. You provide a cell with the constraint and then add the constraint in Solver's dialog box.

There is a conventional graph that is used to visualize constrained optimization problems. It relies on the point of tangency in a contour plot to instantly highlight the solution.

To read a contour plot, remember that it is a top-down view of a 3D surface.

Excel has several 3D surface and contour charts. The chart can be augmented with a variety of drawing objects and text boxes.

Appendix

Figure 6.9 shows a few pros and cons of the two charts. The only con for the contour plot is the prerequisite knowledge needed to read it.

3D Surface		Contour	
Pro	Con	Pro	Con
Clearly a hill	B*=1, C*=4 unclear	B*=1, C*=4 clear	Contour prereq
Global max clear	U* unclear	U* clear	
Constraint clear	C axis unclear	Constraint clear	

Figure 6.9: Comparing 3D surface and 2D contour plots.

For those in the know, a 2D contour plot is preferred because of the way the tangency draws attention to the optimal solution. It is easy to see the values that solve the constrained optimization problem.