Intermediate Microeconomics with Microsoft Excel®

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1.3.1 Initial Solution for the Consumer Choice Problem

Joseph Louis Lagrange, the greatest mathematician of the eighteenth century, was born at Turin on January 25, 1736, and died at Paris on April 10, 1813. … In appearance he was of medium height, and slightly formed, with pale blue eyes and a colourless complexion. In character he was nervous and timid, he detested controversy, and to avoid it willingly allowed others to take credit for what he had himself done.
W. W. Rouse Ball

The *Budget Constraint* shows the consumer's possible consumption bundles.
The standard, linear constraint is $p_1x_1 + p_2x_2 = m$.
There are many other situations, such as subsidies and rationing, which give more complicated constraints with kinks and horizontal/vertical segments.

The *Indifference Map* shows the consumer's preferences.
The standard situation is a set of convex, downward sloping indifference curves.
There are many alternative preferences, such as perfect substitutes and perfect complements.
Preferences are captured by utility functions, which accurately reflect the shape of the indifference curves.

Showing the Initial Solution

Our job is to find the combination (or bundle) that maximizes satisfaction (as described by the indifference map or utility function) given the budget constraint. The answer will be in terms of how much the consumer will buy in units of each good.

The optimal solution is depicted by the canonical graph in Figure 1.3.1.1. This canon is not a cannon as in a weapon that fires projectiles. The word canonical is used here to mean standard, conventional, or orthodox. In economics, a canonical graph is a core, essential graph that is understood by all economists, such as a supply and demand graph.

It is no exaggeration to say that Figure 1.3.1.1 is one of the most fundamental and important graphs in economics. It is the foundation of the Theory of Consumer Behavior and with it we will derive a demand curve.

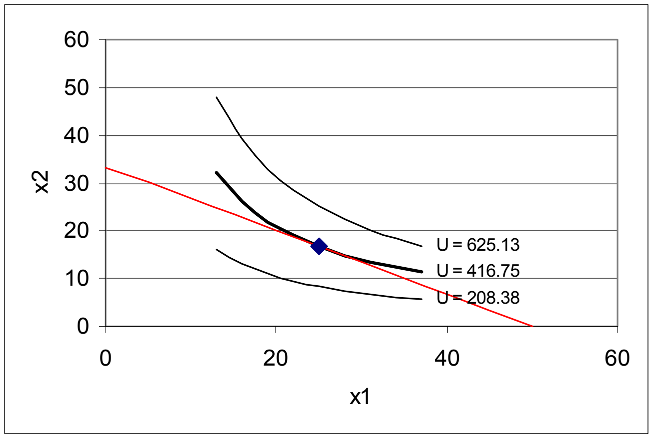


Figure 1.3.1.1: Displaying the optimal solution.

Finding the Initial Solution

There are two ways to find the optimal solution:

1) Analytical methods using algebra and calculus—conventional, paper-and-pencil

2) Numerical methods using a computer (Excel's Solver)

Analytical Approach

Unfortunately, constrained optimization problems are harder to solve than unconstrained problems. The appendix to this chapter offers a short calculus review along with a list of common derivative and algebra rules. If the material below makes little sense, see the appendix and then return here.

Since this is a constrained optimization problem, the analytical approach uses the method developed by Joseph Louis Lagrange.

Lagrange's brilliant idea is based on transforming a constrained optimization problem into an unconstrained problem and then solving using standard calculus techniques. In the process, a new endogenous variable is created. It can have an interesting economic interpretation.

There is a recipe:

1) Rewrite the constraint so that it is equal to zero
2) Form the Lagrangean function
3) Take partial derivatives with respect to x_1, x_2, and λ
4) Set the derivatives equal to zero and solve the system of equations for x_1, x_2, and λ

A Concrete Example

Suppose the consumer has a Cobb-Douglass utility function with exponents equal to 1 and a budget constraint, $2x_1 + 3x_2 = 100$ (which means the price of good 1 is \$2/unit, the price of good 2 is \$3/unit, and income is \$100).

The problem is to maximize utility subject to the budget constraint. This problem is not solved directly. It is first transformed into an unconstrained problem, and then the unconstrained problem is solved.

We apply the recipe developed by Lagrange.

1) Rewrite the constraint so that it is equal to zero

$0 = 100 - 2x_1 - 3x_2$

2) Form the Lagrangean

$$\max_{x_1, x_2, \lambda} L = x_1 x_2 + \lambda(100 - 2x_1 - 3x_2)$$

Note that the Lagrangean function, L, is composed of the original objective function (in this case, the utility function) plus a new variable, λ (the Greek letter lambda) times the rewritten constraint. λ is called the Lagrangean (or Lagrange) multiplier.

3) Take partial derivatives with respect to x_1, x_2, and λ

$$\frac{\partial L}{\partial x_1} = x_2 - 2\lambda$$
$$\frac{\partial L}{\partial x_2} = x_1 - 3\lambda$$
$$\frac{\partial L}{\partial \lambda} = 100 - 2x_1 - 3x_2$$

The derivative used here is a partial derivative, denoted by ∂, which is a lowercase Greek letter d (which is why sometimes δ is used as a symbol for the partial derivative). The partial derivative symbol is often read as the letter d, so the first equation is read as "d L d x-one equals x-two minus two times lambda." It is also common to read the derivative in the first equation as "partial L partial x one."

The partial derivative is a natural extension of the regular derivative. Consider the function $y = 4x^2$. The derivative of *y* with respect to *x* is $dy/dx = 8x$. Suppose, however, that we had a more complicated function, like this: $y = 4zx^2$. This function says that *y* depends on two variables, *z* and *x*. We can explore the rate of change of this function along a single dimension by treating it as a partial function, meaning that we hold all other variables constant. Then the partial derivative of *y* with respect to *x* is $\partial y / \partial x = 8zx$ and the partial derivative of *y* with respect to *z* is $\partial y / \partial z = 4x^2$.

The partial derivative enables us to use the derivative on multivariate functions. Remember to treat other variables as constants when taking a partial derivative.

4) Set the partial derivatives equal to zero and solve the system of equations for x_1, x_2, and λ

$$\frac{\partial L}{\partial x_1} = x_2 - 2\lambda = 0$$
$$\frac{\partial L}{\partial x_2} = x_1 - 3\lambda = 0$$
$$\frac{\partial L}{\partial \lambda} = 100 - 2x_1 - 3x_2 = 0$$

There are many ways to solve this system of equations, which are known as the first-order conditions. A common strategy involves moving the λ terms to the right-hand side and then dividing the first equation by the second one, like this.

$$x_2 = 2\lambda$$
$$x_1 = 3\lambda$$
$$\frac{x_2}{x_1} = \frac{2\lambda}{3\lambda}$$

The λ terms then cancel out, leaving us with two equations (the one above and the third equation from the original three first-order conditions) and two unknowns (x_1 and x_2).

$$\frac{x_2}{x_1} = \frac{2}{3}$$
$$100 - 2x_1 - 3x_2 = 0$$

The top equation has a nice economic interpretation. It says that, at the optimal solution, the MRS (slope of the indifference curve) must equal the price ratio (slope of the budget constraint).

From the top equation, we can solve for x_2.

$$x_2 = \frac{2}{3}x_1$$

We can then substitute this value into the second equation to get the optimal value of x_1.

$$
\begin{aligned}
&100 - 2x_1 - 3\left[\frac{2}{3}x_1\right] = 0 \\
&100 - 2x_1 - 2x_1 = 0 \\
&100 = 4x_1 \\
&x_1^* = 25
\end{aligned}
$$

Then we substitute this value into the expression for x_2 to get the optimal value of x_2.

$$
\begin{aligned}
&x_2 = \frac{2}{3}[25] \\
&x_2^* = 16\,{}^2\!/_3
\end{aligned}
$$

The asterisk is used to represent the optimal solution for a choice variable. This consumer should buy 25 units of good 1 and 16 and 2/3 units of good 2 in order to maximize satisfaction given the budget constraint.

We can use either equation 1 or 2 (from the original first order conditions) to find the optimal value of lambda. Either way, we get $\lambda^* = 8\ 1/3$.

For many optimization problems, we would be interested in finding the value of the maximum by evaluating the objective function (in this case the utility function) at the optimal solution. But recall that utility is measured only up to an ordinal scale and the actual value of utility is irrelevant. We want to maximize utility, but we do not care about its actual maximum value.

Numerical Approach

Instead of calculus (via the method of Lagrange) and pencil-and-paper, we can use numerical methods to find the optimal solution.

We have to set up the problem in Excel, carefully organizing things into a Goal, Endogenous Variables, Exogenous Variables, and Constraint; then use Excel's Solver to get the solution.

Step Open the Excel workbook OptimalChoice.xls and read the *Intro* sheet, then go to the *OptimalChoice* sheet to see how the numerical approach can be used to solve this problem.

Figure 1.3.1.2 reproduces the display when you first arrive at the *OptimalChoice* sheet.

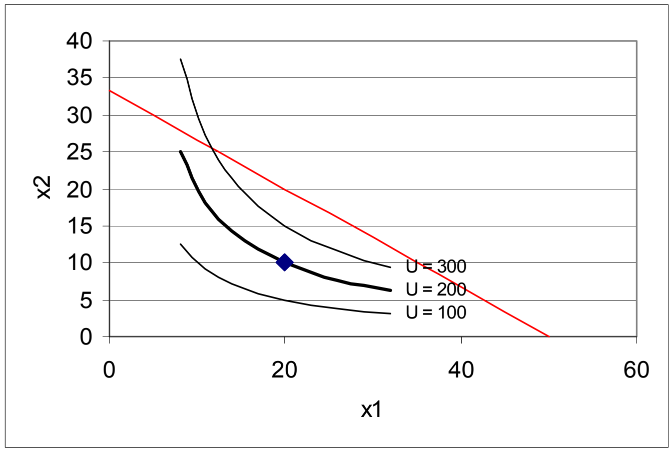
$$\max_{x_1, x_2} \; u(x_1, x_2) = x_1^c x_2^d$$
$$\text{s.t. } m \geq p_1 x_1 + p_2 x_2$$

Goal	
max Utility	200

Endogenous Variables	
x1	20
x2	10

Exogenous Variables		
p1	2	price of x1
p2	3	price of x2
m	100	income
c	1	exponent for x1
d	1	exponent for x2

Constraint	30	income left over



-p1/p2	MRS at x1, x2
-0.667	-0.500

Figure 1.3.1.2: The initial display in the *OptimalChoice* sheet.
Source: OptimalChoice.xls!OptimalChoice.

Notice how the sheet is organized by the three components of the optimization problem, goal, endogenous and exogenous variables. The constraint cell displays how much of the consumer's budget remains available for buying goods. The consumer in Figure 1.3.1.2 is not using all of the income available so we know satisfaction cannot be maximized at the point 20,10.

step

Let's have the consumer buy x_2 with the remaining \$30. At the \$3/unit, 10 additional units of x_2 can be purchased. Enter 20 in the x_2 cell (B13) and hit the Enter key. The chart refreshes to display the point 20,20, which is on the budget constraint, and draws three new indifference curves.

Although 20,20 does exhaust the available income, it is not the optimal solution. The display at the bottom reveals the MRS does not equal the price ratio.

In absolute value, the MRS > p_1/p_2, in other words, the slope of the indifference curve at that point is greater than the slope of the budget constraint.

The consumer cannot change the slope of the budget constraint, but the MRS can be altered by changing the combination of goods purchased. This consumer needs to lower the MRS (in absolute value) to make the two equal. This can be done by crawling down the budget constraint.

If the consumer buys 10 more of good 1 (so 30 units of x_1 total), consumption of x_2 must fall by 6 and 2/3 units to 13.33 (repeating, of course).

Enter 30 in cell B12 and 13.33 in B13. (You can enter "=13 + 1/3" if you want more precision, but Excel cannot perfectly accurately represent a repeating decimal.) Now you are on the other side of the optimal solution. The MRS is less than the price ratio.

You could, of course, continue adjusting the cells, but there is a faster way.

Click Data and click Solver (grouped under the Analysis tab) or execute Tools: Solver in older versions of Excel to bring up the Solver Parameters dialog box (displayed in Figure 1.3.1.3).

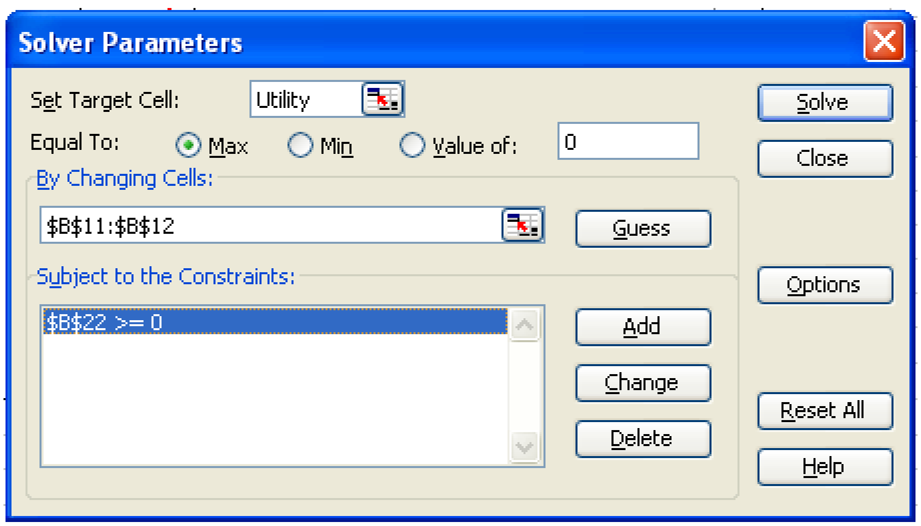


Figure 1.3.1.3: Excel's Solver.

If you do not have Solver available as a choice, bring up the Add-in Manager dialog box and make sure that Solver is listed and checked. If Solver is not listed, you must install it from the Office CD or download from <www.solver.com>.

Notice how Excel's Solver includes information on the objective function (the target cell), the choice variables (the changing cells), and the budget constraint.

All of the information has been entered into the Solver Parameters dialog box so you simply click the Solve button.

Excel's Solver works by trying different combinations of x_1 and x_2 and evaluating the improvement in the target cell, while meeting the constraint. When it cannot improve very much more, it figures it has found the answer and displays a message as shown in Figure 1.3.1.4.

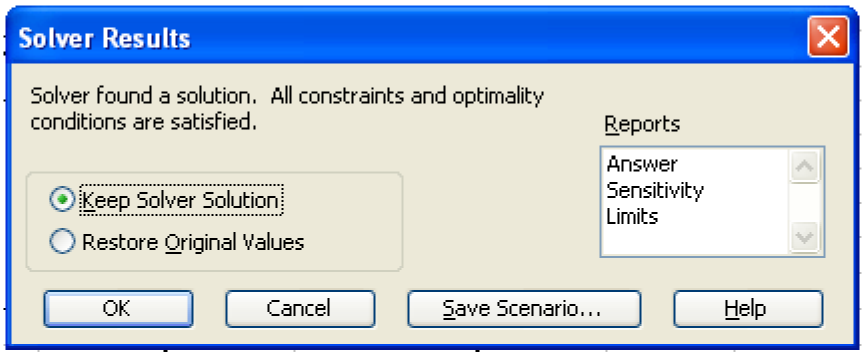


Figure 1.3.1.4: Solver reports success.

While Solver gets the right answer in this problem, we will see in future applications that Solver is not perfect and does not deserve blind trust.

Click the Sensitivity option under Reports and click OK, Excel puts down the Solver solution into cells B12 and B13. It also inserts a new sheet into the workbook with the Sensitivity Report.

Click on cells B12 and B13. Notice that Excel did not get exactly 25 and 16 and 2/3. It got extremely close and you can certainly interpret the result as confirming the analytical solution, but Solver's output will require interpretation.

You can confirm that Excel's Sensitivity Report gives the same absolute value, 8.33, for the Lagrangean multiplier that we found via the Lagrangean method. In later chapters, we will explain what this means. For now, we simply note that the Excel results agreed with the Lagrangean method.

You might notice that Excel reports a Lagrangean Multiplier value of – 8.33. It turns out that we ignore the sign of λ^*. If we set up the Lagrangean as the objective function *minus* lambda times the constraint or rewrite the constraint as $0 = 2x_1 + 3x_2 - 100$ (instead of $0 = 100 - 2x_1 - 3x_2$), we would get a negative value for λ^*. The way we write the constraint or whether we add or subtract the constraint is arbitrary, so we ignore the sign of λ^*.

Unlike the sign, the magnitude of λ^* can be meaningful. Since utility is not cardinal, λ^* does not have an interesting economic interpretation in this problem, but we will see applications where the value of λ^* is useful.

Using Analytical and Numerical Methods to Find the Optimal Solution

There are two ways to solve optimization problems.

The traditional way uses pencil-and-paper, derivatives and algebra. The Lagrangean Method is used to solve constrained optimization problems, like the consumer's choice problem.

Advances in computers have led to the creation of numerical methods to solve optimization problems. Excel's Solver is an example of a numerical algorithm that can be used to find optimal solutions.

In the chapters that follow, we will continue to use both analytical and numerical approaches. You will see that neither method is perfect and both have strengths and weaknesses.

Exercises

Open Word and answer the following questions. Save the document and print it when you are done.

The utility function, $U = 10x - 0.1x^2 + y$, has a quasilinear functional form. Use this utility function to answer the questions below.

1) Suppose the budget line is $100 = 2x + 3y$. Use the analytical method to find the optimal solution. Show your work.
2) Suppose the consumer considers the bundle 0,33.33, buying no x and spending all income on y. Use the MRS compared to the price ratio logic to explain what the consumer will do and why.
3) Consider the parameters in the utility function, a, b, c, and d ($U = ax - bx^c + dy$). If a increases, what happens to the optimal consumption of x^*? Explain how you arrived at your answer.

References

The epigraph is from page 421 of W.W. Rouse Ball's *A Short Account of the History of Mathematics* (first published in 1888). Of course, there are many books on the history of mathematics, but this classic is fun and easy to read. It mixes stories about people with real mathematical content.

This entire book (and many others) is freely available at <books.google.com>. You can read it online or download it as a pdf file.

Appendix: Derivatives and Optimization

A derivative is a mathematical expression that tells you how y in a function $y = f(x)$ changes given an infinitesimally small change in x. Graphically, it is the slope, or rate of change, of the function at that particular value of x.

Linear functions have a constant slope and, therefore, a constant value for the derivative. For the linear function $y = 6 + 3x$, the derivative of y with respect to x is written dy/dx (pronounced "d y d x") and its value is 3. This tells you that every time the x variable goes up, the y variable goes up 3-fold. So, if x increases by 1 unit, y will increase by 3 units. This is easy to see in Figure 1.3.1.5.

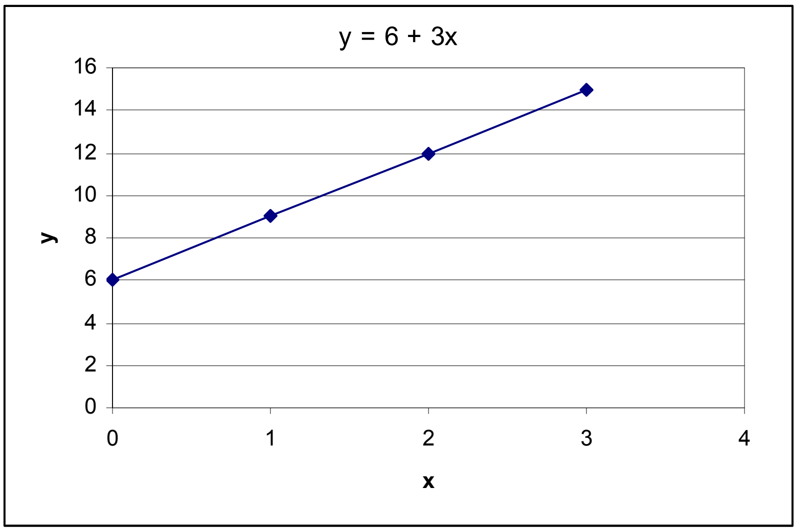


Figure 1.3.1.5: A linear function.

For linear functions, the size of the change in x does not affect the rate of change. So, if x increases by 2 units (say from 1 to 3), then y increases by 6 units (from 9 to 15) and the rate of change, defined as the change in y divided by the change in x, remains 3.

Another simple property of linear functions is that the slope remains the same no matter the value of x. In Figure 1.3.1.5, the slope is 3 when you increase x from 1 to 2 or from 3.000 to 3.001.

An easy way to tell if a function is linear is to compute the derivative and check to see if x appears in the derivative. With $y = 6 + 3x$, $\frac{dy}{dx} = 3$ and x does not appear in the derivative. A mathematician would say, "In this case, the slope is constant so y is linear in x."

Non-linear functions have a changing slope and, therefore, a derivative that takes on different values at different values of x. Consider the function $y = 4x - x^2$. Its derivative is $\frac{dy}{dx} = 4 - 2x$. Notice that the derivative has x in it. This means the function is non-linear.

Because it is non-linear, the size of the change in x affects the rate of change and the rate of change depends on the value of x. Figure 1.3.1.6 graphs this function.

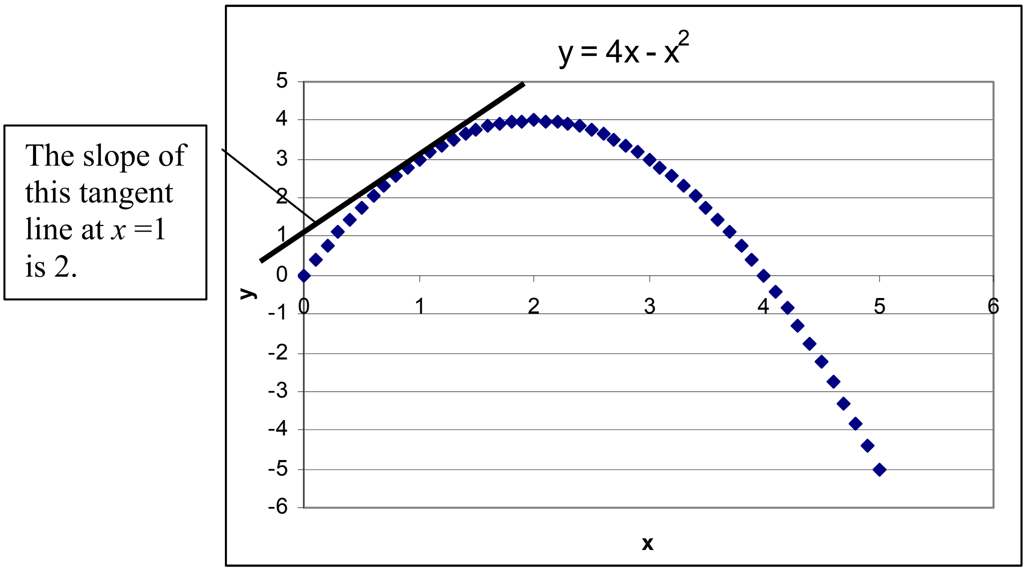


Figure 1.3.1.6: A non-linear function with a tangent line at $x = 1$.

With a non-linear function, the size of the change in x leads to different measures of the slope. The change in y from $x = 1$ to $x = 2$ is 1 (because we move from $y = 3$ to $y = 4$ as we increase x by 1). If we increase x by 0.1 (from 1 to 1.1), the $\frac{\Delta y}{\Delta x} = \frac{3.19 - 3}{1.1 - 1} = 1.9$. By taking a smaller change in x, we get a different measure of the rate of change.

If we compute the rate of change via the derivative, by evaluating $4 - 2x$ at $x = 1$, we get 2. The derivative computes the rate of change for an infinitesimally small change in x. The smaller the change in x, the closer $\frac{\Delta y}{\Delta x}$ gets to $\frac{dy}{dx}$. The derivative is based on the rate of change of the slope of the tangent line, as shown in Figure 1.3.1.6.

Figure 1.3.1.6 makes clear that the slope, or rate of change, of the function varies along the curve. The rate of change is 2 at $x = 1$ and -2 at $x = 3$ (found by plugging 3 into the derivative and computing $4 - 2[3]$). The minus means the function is downward sloping.

Optimizing with the Derivative

An optimization problem typically requires you to find the value of an endogenous variable (or variables) that maximizes or minimizes a particular objective function. We can use derivatives to find the optimal solution. This is called an analytical approach.

Figure 1.3.1.6 shows that the maximum of the function is where the slope is zero. By finding the flat spot, we find the top.

By solving for the value of x where $\frac{dy}{dx}=0$, we find the optimal solution. For $y=4x-x^2$, this is easy. We set the derivative equal to zero and solve for x^*:

$$\begin{aligned}&\frac{dy}{dx}=4-2x^*=0\\&4=2x^*\\&x^*=2\end{aligned}$$

The equation that you make when you set the first derivative equal to zero is called the *first-order condition*. The first-order condition (FOC) is different from the derivative because the derivative by itself is not equal to anything—you can plug in any value of x and the derivative expression will pump out an answer that tells you whether and by how much the function is rising or falling at that point.

A *reduced-form* is the answer that you get when the derivative is set equal to zero and solved for the optimal solution. It may be a number or a function of exogenous variables. It cannot have any endogenous variables in the expression. Sometimes, you cannot solve explicitly for x^*. We say there is no closed form solution in these cases. The solution may exist (and numerical methods may be used to find it), but we cannot express the answer as an equation.

The second derivative is simply the derivative of the first derivative. It tells you the slope of the slope function. For example, if a function has a constant slope, we saw that its first derivative is a constant value (like 3 in the first example). The second derivative of the function tells you how the slope changes when x changes. Well, since the slope is unchanging, the second derivative would be zero.

Second derivatives are useful for the following reason: when you find the value of the endogenous variable that makes the first derivative equal to zero, the point that you have located could be either a maximum or a minimum. If you want to be sure which one you have found, you can check out the second derivative. For $y=4x-x^2$, the first derivative is $4-2x$ and the second derivative is -2. Since the second derivative is negative, we know that our flat spot at $x=2$ is a maximum and not a minimum.

In summary, derivatives are used to measure the rate of change of a function. If we set a derivative equal to zero, we are trying to find an optimal solution by finding where the function is flat. This appendix concludes with a short list of common rules for taking derivatives and other useful math facts.

Rules for Taking Derivatives

A derivative can be computed by directly applying the definition—i.e., taking the limit of the change in x as it approaches zero and determining the change in y. Fortunately, however, there is an easier way. Differentiation rules have been developed that make it much less tedious to take a derivative. Most calculus books have inside covers that are full of rules. Many students never grasp that these rules are actually shortcuts. Here is a short list, with special emphasis on those used in economics.

Let x be the variable and a be a constant.

General Rule	Example of its Application
$\frac{d}{dx}(x) = 1$	
$\frac{d}{dx}(ax) = a$	$\frac{d}{dx}(4x) = 4$
$\frac{d}{dx}(a) = 0$	$\frac{d}{dx}(4) = 0$
$\frac{d}{dx}(x^a) = ax^{a-1}$	$\frac{d}{dx}(x^4) = 4x^3$
$\frac{d}{dx}(a\ln x) = \frac{a}{x}$	$\frac{d}{dx}(4\ln x) = \frac{4}{x}$

Chain rule: derivative of the whole thing times the derivative of the inside

$\frac{d}{dx}((f(x))^a) = a(f(x))^{a-1} f'(x)$	$\frac{d}{dx}((x^2)^3) = 3x^2 2x$

Product Rule: derivative of the first times the second + the first times the derivative of the second

$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$	$\frac{d}{dx}((2x+3)(4x)) = (2)(4x) + (2x+3)(4)$

When you take a derivative of a function with respect to a variable, you apply the rules to the different parts of the function. For example, if $y = 4x - x^2$, then you apply the $\frac{d}{dx}(ax) = a$ rule to the $4x$ part of the function, getting 4. You apply the $\frac{d}{dx}(x^a) = ax^{a-1}$ rule to the $-x^2$ term and get $-2x$. Thus, the derivative of y with respect to x is $\frac{dy}{dx} = 4 - 2x$.

Laws of Exponents

We end this appendix with a short list of algebra rules relating to legal operations on exponents. We will use these rules often to find optimal solutions and reduce complicated expressions to simpler final answers.

General Rule	Example of its application
$x^0 = 1$	
$x^{-a} = \frac{1}{x^a}$	$x^{-1/2} = \frac{1}{\sqrt{x}}$
$x^a x^b = x^{a+b}$	$x^2 x^3 = x^5 \Rightarrow 2^2 2^3 = 2^5 = 32$
$\frac{x^a}{x^b} = x^{a-b}$	$\frac{x^5}{x^3} = x^2 \Rightarrow \frac{2^5}{2^3} = 2^2 = 4$
$(xy)^a = x^a y^a$	$(xy)^2 = x^2 y^2 \Rightarrow (2 \cdot 3)^2 = 2^2 3^2 = 36$
$(x^a)^b = x^{ab}$	$(x^2)^3 = x^6 \Rightarrow (2^2)^3 = 2^6 = 64$

1.3.2 More Practice and Understanding Solver

The methods of mathematics apply as soon as spatial or numerical attributes are associated with our phenomena, as soon as objects can be located by points in space and events described by properties capable of indication or measurement in numbers.
R. G. D. Allen

We know there are two approaches to solving optimization problems:

1) Analytical methods using algebra and calculus (conventional, paper-and-pencil using the Lagrangean Method)
   The idea is to transform the consumer's constrained optimization problem into an unconstrained problem and then solve it using standard unconstrained calculus techniques—i.e., take derivatives, set equal to zero, and solve the system of equations.

2) Numerical methods using a computer (Excel's Solver)
   Set up the problem in Excel, carefully organizing things into a Goal, Endogenous Variables, Exogenous Variables, and Constraint; then use Excel's Solver (Tools: Solver). Use the Sensitivity Report in the Solver Results dialog box to get λ*.

We will practice applying the analytical method and begin learning about how Excel's Solver actually works.

Quasilinear Utility Practice Problem

A utility function that is composed of a non-linear function of one good plus a linear function of the other good is called a quasilinear functional form. It is quasi, or sort of, linear because one good increases utility in a linear fashion and the other does not.

Below are a general example and a more specific example of quasilinear utility.

$$u(x_1, x_2) = k = v(x_1) + x_2$$
$$u(x_1, x_2) = (x_1)^c + x_2, \text{where } c \neq 1$$

If $c < 1$, then the quasilinear utility function says that utility increases at a decreasing rate as x_1 increases, but utility increases at a constant rate as x_2 increases.

The optimization problem is to maximize utility subject to the usual budget constraint.

First, we solve the general version of this problem via analytical methods.

1) Rewrite the constraint so that it is equal to zero

$0 = m - p_1x_1 - p_2x_2$

2) Form the Lagrangean

$$\max_{x_1,x_2,\lambda} \mathrm{L} = x_1^c + x_2 + \lambda(m - p_1x_1 - p_2x_2)$$

Note that the Lagrangean function, L, has the quasilinear utility function plus the Lagrangean multiplier, λ, times the rewritten constraint.

Unlike the concrete problem in the last chapter, which used numerical values, this is a general problem with letters indicating exogenous variables. General problems, without numerical values for exogenous variables, are harder to solve because we have to keep track of many variables. If the solution can be written as a function of the exogenous variables, however, it is often easy to see how an exogenous variable will affect the optimal solution.

3) Take partial derivatives with respect to x_1, x_2, and λ

$$\frac{\partial L}{\partial x_1} = cx_1^{c-1} - p_1\lambda$$
$$\frac{\partial L}{\partial x_2} = 1 - p_2\lambda$$
$$\frac{\partial L}{\partial \lambda} = m - p_1x_1 - p_2x_2$$

Remember that the partial derivative treats other variables as constants. Thus, the partial derivative of the quasilinear utility function with respect to x_1 has no x_2 variable in it.

4) Set the partial derivatives equal to zero and solve the system of equations for x_1, x_2, and λ

$$\frac{\partial L}{\partial x_1} = cx_1^{c-1} - p_1\lambda = 0$$
$$\frac{\partial L}{\partial x_2} = 1 - p_2\lambda = 0$$
$$\frac{\partial L}{\partial \lambda} = m - p_1x_1 - p_2x_2 = 0$$

We use the same solution method as before, moving the lambda terms to the right hand side and then dividing the first equation by the second, which allows us to cancel the lambda terms.

$$cx_1^{c-1} = p_1\lambda$$
$$1 = p_2\lambda$$
$$\frac{cx_1^{c-1}}{1} = \frac{p_1\lambda}{p_2\lambda}$$
$$\frac{cx_1^{c-1}}{1} = \frac{p_1}{p_2}$$

By cancelling the lambda terms, we have reduced the three equation, three unknown system to two equations with two unknowns.

$$\frac{cx_1^{c-1}}{1} = \frac{p_1}{p_2}$$
$$m - p_1x_1 - p_2x_2 = 0$$

Remember that not all variables are the same. The endogenous variables, the unknowns, are x_1 and x_2. The other letters are exogenous variables.

From the first equation, we can solve for the optimal quantity of good 1.

$$\frac{cx_1^{c-1}}{1} = \frac{p_1}{p_2}$$
$$cx_1^{c-1} = \frac{p_1}{p_2}$$
$$x_1^{c-1} = \frac{p_1}{cp_2}$$
$$x_1^* = \left(\frac{p_1}{cp_2}\right)^{\frac{1}{c-1}}$$

Notice that we used the rule that $\left(x^a\right)^b = x^{ab}$. Since we wanted to solve for x_1, we raised both sides to the $1/_{c-1}$ power so that c-1 times $1/_{c-1}$ would give 1.

Usually, when we have the MRS equal to the price ratio, we need to solve for one of the x variables in terms of the other and substitute it into the budget constraint. However, a property of the quasilinear utility function is that the MRS only depends on x_1, thus by solving for x_1, we get the reduced-form equation.

To get x_2, we simply substitute x_1 into the budget constraint and solve for x_1.

$$m - p_1\left[\left(\frac{p_1}{cp_2}\right)^{\frac{1}{c-1}}\right] - p_2 x_2 = 0$$

$$x_2^* = \frac{m}{p_2} - \frac{p_1}{p_2}\left(\frac{p_1}{cp_2}\right)^{\frac{1}{c-1}}$$

It is a bit messy, but it is the answer. We have an expression for the optimal amount of x_2 that is a function of exogenous variables alone.

To get the optimal value of lambda, we can use the second first order condition, which simply says that $\lambda^* = 1/p_2$. If you use the first condition, substituting in the value for optimal x_1 it will take a little work, but you will get the same result.

Practice with the MRS = p_1/p_2 Logic

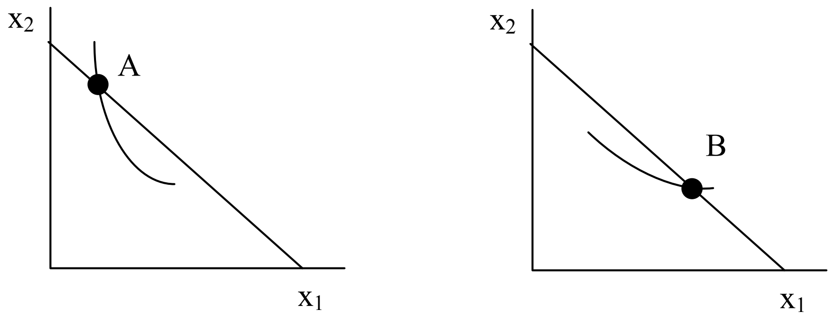
Economists stress marginal thinking. The idea is that, from any position, you can move and see how things change. If there is improvement, continue moving. The optimal solution is on a flat spot, where improvement is impossible.

When we move the lambda terms over to the right hand side and divide equation one by equation two, we get a crucial statement of the fact that improvement is impossible and we are optimizing.

The familiar MRS = price ratio expression, along with the third first order condition, which says that the consumer must be on the budget line (exhausting all income), is a mathematical way of describing marginal thinking.

The MRS condition tells us that if the MRS is not equal to the price ratio, there are two possibilities, depicted in Figure 1.3.2.1.

In Panel A, the slope of the indifference curve at point A is greater than the slope of the budget line (in absolute value). This consumer should crawl down the budget line, reaching higher indifference curves, until the MRS equals the price ratio. At this point, the slope of the indifference curve will exactly equal the slope of the budget line and the consumer's indifference curve will just touch the budget line.



Panel A: MRS $> p_1/p_2$ Panel B: MRS $< p_1/p_2$

Figure 1.3.2.1: MRS ≠ price ratio.

In Panel B, the story is the same, but reversed. The slope of the indifference curve at point B is less than the slope of the budget line. This consumer should crawl up the budget line, reaching higher indifference curves, until the MRS equals the price ratio. At this point, the slope of the indifference curve will exactly equal the slope of the budget line and the consumer's indifference curve will just touch the budget line.

Numerical Approach to Quasilinear Practice Problem

Open the Excel workbook OptimalChoicePractice.xls and read the *Intro* sheet, then go to the *QuasilinearChoice* sheet to see how the numerical approach can be used to solve this problem.

The consumer cannot afford the bundle 5, 20. If she buys 5 units of x_1, what's the maximum x_2 she can buy?

Enter this amount in cell B12.

The chart updates and shows that the consumer is now on the budget line. In addition, the constraint cell, B21, is now zero.

Without running Solver or doing any calculations at all, is she maximizing at 5, 13? No. It's hard to see on the chart if the indifference curve is cutting the budget line, but the information below the chart shows that the MRS is not equal to the price ratio. That tells you that the indifference curve is, in fact, not tangent to the budget line so the consumer is not optimizing. Since the MRS is greater than the price ratio (in absolute value) we also know that the consumer should buy more x_1 and less x_2, moving down the budget line until the marginal condition is satisfied.

Step Run Solver. Select the Sensitivity Report to get λ*.

We can compare Solver's result to our analytical result. Recall that

$$x_1^* = \left(\frac{p_1}{cp_2}\right)^{\frac{1}{c-1}}$$

$$x_2^* = \frac{m}{p_2} - \frac{p_1}{p_2}\left(\frac{p_1}{cp_2}\right)^{\frac{1}{c-1}}$$

Step Create formulas in Excel to compute these two solutions (using cells C11 and C12 would make sense).

Step Create formulas in cells D11 and D12 that compute the difference between the numerical and analytical answers.

You should discover that Excel's Solver is slightly off the computed analytical result. There are two reasons for the discrepancy.

1) Excel cannot display the algebraic result to an infinite number of decimal places. If the solution is a repeating decimal or irrational number, Excel cannot handle it. Even if the number can be expressed as a decimal, for example, one-half, Excel applies finite-precision, binary computations and, therefore, it may induce rounding error.

2) Excel's Solver often misses the exactly correct answer by small amounts. Solver has a convergence criterion (that you can set via the Options button in the Solver Parameters dialog box) that determines when it stops hunting for a better answer. Figure 1.3.2.2 offers a graphical representation of Solver's algorithm in a one-variable case.

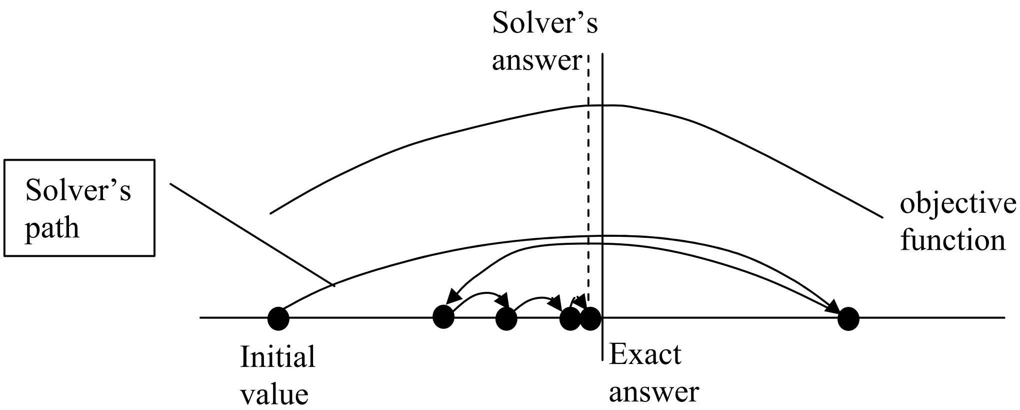


Figure 1.3.2.2: Solver in action.

The stylized graph (which means it represents an idea without using actual data) in Figure 1.3.2.2 shows that Solver works by trying different values and seeing how much improvement occurs. The path of the choice variable (on the *x* axis) is determined by Solver's internal optimization algorithm. By default, it uses Newton's Method, but you can choose an alternative by clicking the Options button in the Solver dialog box.

When Solver takes a step that improves the value of the objective function by very little, determined by the Convergence criterion (adjustable via the Options button), it stops searching and announces success. In Figure 1.3.2.2, Solver is missing the optimal solution by a little bit because the objective function is almost flat at the top. Solver cannot distinguish additional improvement.

When we say that the analytical method agrees with Solver, we do not mean that the two methods exactly agree, but simply that they correspond, in a practical sense. If Solver is off the exact answer in the 15th decimal place, that is agreement, for all practical purposes.

In the quasilinear utility function example, we would conclude that Solver and the calculus agree because they are very close.

Now, let's learn that Solver is not perfect.

Step Start from $x_1 = 1$, $x_2 = 20$ to see an example of a miserable result. After setting the cells to 1 and 20, run Solver. What happens?

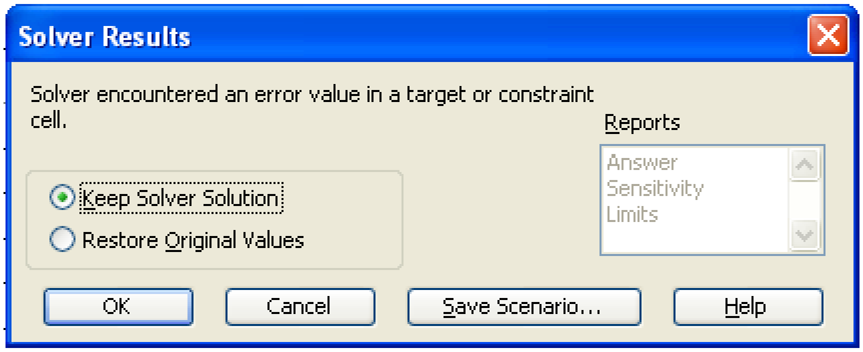


Figure 1.3.2.3: A miserable result.

A *miserable result* (an actual, technical term in the numerical methods literature) occurs when an algorithm reports that it cannot find the answer or displays an obviously erroneous solution. Figure 1.3.2.3 displays an example of a miserable result. Solver is clearly announcing that it cannot find an answer.

If you look carefully at the spreadsheet, you will see that Solver blew up when it tried a negative value for x_1. The objective function cell, B7, is displaying the error #NUM! because Excel cannot take the square root of a negative number.

When Solver can't find an answer, there are three basic strategies to fix the problem:

1) Try different initial values (in the changing cells).
   - If you know roughly where the solution lies, start near it.
   - Always avoid starting from zero or a blank cell.
2) Add more structure to the problem.
   - Include non-negativity constraints on the endogenous variables, if appropriate.
   - In the case of consumer theory, if you know the buyer will be on the budget constraint, use an equality constraint.
3) Completely reorganize the problem.
   - Instead of directly optimizing, you can put Solver to work on equations that must be met.
   - In this problem, you know that MRS = p1/p2 is required. You could create a cell that is the difference between the MRS and the price ratio and have Solver find the values of the choice variable that force this cell to equal zero.

Perfect Complements Practice Problem

Recall that L-shaped indifference curves represent perfect complements, which are reflected via the following mathematical function:

$$u(x_1, x_2) = \min\{ax_1, bx_2\}$$

Suppose $a = b = 1$. Suppose the budget line is $50 = 2x_1 + 10x_2$.

We want to solve this problem analytically.

The first thing to realize is that the Lagrangean method cannot be applied. The function is not differentiable at the corner of the L.

The Lagrangean method, however, is not the only analytical method available. Figure 1.3.2.4 shows that when $a = b = 1$, the optimal solution must lie on a ray from the origin with slope +1.

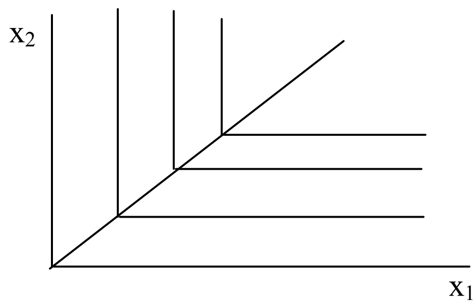


Figure 1.3.2.4: The optimal solution line with perfect complements.

The optimal solution has to be on the corner of the L-shaped indifference curves because a non-corner point (on either the vertical or horizontal part of the indifference curve) implies the consumer is spending money on more of one of the goods without getting any additional satisfaction.

The equation of the optimal solution line is simple: $x_2 = x_1$.

.

We can combine this equation with the budget constraint to find the optimal solution. The two equation, two unknown system can be solved easily by substitution.

$$\left.\begin{array}{l} x_2 = x_1 \\ 50 = 2x_1 + 10x_2 \end{array}\right\} \Rightarrow 50 = 2x_1 + 10[x_1] \Rightarrow 50 = 12x_1 \Rightarrow x_1^* = 4\,{}^1\!/_6.$$

Of course, we know $x_2 = x_1$ so $x_2^* = 4\,{}^1\!/_6$.

Step Proceed to the *PerfectComplements* sheet to see that Excel's Solver can also solve this problem. Notice that Excel's Solver can be used to generate a value for the Lagrangean multiplier (via the Sensitivity Report) even though we did not use the Lagrangean method.

As with the previous problem (with quasilinear utility), we find that Solver and the analytical approach substantially agree. The answer is a repeating decimal, so Excel cannot get the exact answer, but it comes extremely close.

Now, let's learn that Solver can really misbehave.

Start from $x_1 = 1$, $x_2 = 1$ to see an example of a disastrous result. After setting the cells to 1 and 1, run Solver. What happens?

Solver reports a successful outcome, but the answer is 1,1 and we know the right answer is about 4.167, 4.167.

This is an example of a *disastrous result* which occurs when an algorithm reports that it has found the answer, but it is wrong. There is no obvious error and the user may well accept the answer as true.

Disastrous results include an element of interpretation. In this case, we might notice that 1, 1 is way inside the budget constraint and, therefore, the algorithm has failed. A truly disastrous result occurs when there is no way to independently test or verify the algorithm's wrong answer.

Miserable and disastrous results are well-defined and understood, technical terms in the mathematical literature on numerical methods. Disastrous results are much more dangerous than miserable results. The latter are frustrating because the computer cannot provide an answer, but disastrous results lead the user to believe an answer that is actually wrong. In the world of numerical optimization, they are a fact of life. Numerical methods are not perfect. You should not completely trust any optimization algorithm.

Understanding Solver—Be Skeptical

This chapter enabled practice solving the consumer's constrained optimization problem with two different utility functions, a quasilinear function and perfect complements. In both cases, we found that Excel's Solver agreed, practically speaking, with the analytical method.

In addition, Excel's Solver was explored in detail. It works by evaluating the objective function for different values of the choice variables. It can fail by reporting that it cannot find a solution (called a miserable result) or—even worse—by reporting an incorrect answer (which is a disastrous result).

It is easy to believe that a result displayed by a computer is guaranteed to be correct. Do not be careless and trusting—numerical methods can and do fail, sometimes spectacularly.

Exercises

Open Word and answer the following questions. Save the document and print it when you are done.

1) In the quasilinear example in this chapter, use the first equation in the first order conditions to find λ^*. Show your work.
2) Use analytical methods to find the optimal solution for the same perfect complements problem as presented in this chapter, except that $a = 4$ and $b = 1$. Show your work.
3) Draw a graph (using Word's Drawing Tools) of the optimal solution for the previous question.
4) Use Excel's Solver to confirm that you have the correct answer. Take a picture of the cells that contain your goal, endogenous variables, and exogenous variables.

References

As economics became more mathematical, a new course was born, Math Econ. The course needed books and R. G. D. Allen's *Mathematical Analysis for Economists* (first published in 1938) became a classic textbook. As E. Schneider, a reviewer, said, "This book fills a long-felt want. At last we possess a book which presents the mathematical apparatus necessary to a serious study of economics in a form suited to the needs of the economist." (*The Economic Journal*, Vol. 48, No. 191. (Sep., 1938), p. 515). The epigraph is from page 2 of *Mathematical Analysis for Economists*, as Allen discusses how and why mathematics can be applied to the study of economics.

1.3.3 Food Stamps

Tastes are the unchallengeable axioms of a man's behavior;
he may properly (usefully) be criticized for inefficiency in
satisfying his desires, but the desires themselves are *data*.
George J. Stigler and Gary S. Becker

This chapter applies the consumer choice model to a real-world example. We will see that the model can be used to explain why someone would sell food stamps. We also tackle an important policy question, "If cash is better than food stamps, why does the food stamp program exist?"

About the US Food Stamp Program

The Food Stamp Program is run by the Department of Agriculture (USDA). They say,

> The Food Stamp Program serves as the first line of defense against hunger. It enables low-income families to buy nutritious food with Electronic Benefits Transfer (EBT) cards. Food stamp recipients spend their benefits to buy eligible food in authorized retail food stores.
> <www.fns.usda.gov/fsp>

Before EBT cards, recipients were given a booklet with different denominations of paper food coupons that were torn off and used as bills. Figure 1.3.3.1 shows a typical food stamp booklet on the cover of a USDA publication.

Figure 1.3.3.1: Old style food stamps.

Today, recipients are given an EBT card that is like a credit card. The recipient swipes it and the amount is deducted from the account. Each month, more money is added.

Since its inception in 1969, food stamps have been used only to purchase food. You cannot buy alcoholic beverages or tobacco, prepared hot food, or non-food items such as laundry detergent.